

Parameter space for families of Parabolic-like mappings

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Abstract

In this paper we study analytic families of degree 2 parabolic-like mappings $(f_\lambda)_{\lambda \in \Lambda}$. We prove that the corresponding family of hybrid conjugacies induces a continuous map $\chi : \Lambda \rightarrow \mathbb{C}$, which associates to each $\lambda \in \Lambda$ the multiplier of the fixed point of the hybrid equivalent rational map P_λ . We prove that, under suitable conditions, the map χ restricts to a ramified covering from the connectedness locus of $(f_\lambda)_{\lambda \in \Lambda}$ to the connectedness locus $M_1 \setminus \{1\}$.

1 Definition

Definition 1.1. Let $\Lambda \subset \mathbb{C}$, $\Lambda \approx \mathbb{D}$ and let $\mathbf{f} = (f_\lambda : U'_\lambda \rightarrow U_\lambda)_{\lambda \in \Lambda}$ be a family of parabolic-like mappings. Set $\mathbf{U}' = \{(\lambda, z) \mid z \in U'_\lambda\}$, $\mathbf{U} = \{(\lambda, z) \mid z \in U_\lambda\}$, $\Omega'_f = \{(\lambda, z) \mid z \in \Omega'_\lambda\}$, $\Omega_f = \{(\lambda, z) \mid z \in \Omega_\lambda\}$ and $f(\lambda, z) = (\lambda, f_\lambda(z))$. Then \mathbf{f} in an analytic family of parabolic-like maps if the following conditions are satisfied:

1. $\mathbf{U}', \mathbf{U}, \Omega'_f$ and Ω_f are homeomorphic over Λ to $\Lambda \times \mathbb{D}$;
2. the projection from the closure of Ω'_f in \mathbf{U} to Λ is proper;
3. the map $f : \mathbf{U}' \rightarrow \mathbf{U}$ is complex analytic and proper. In particular $f(\lambda, z)$ is continuous and holomorphic in (λ, z) ;
4. for each $\lambda \in \Lambda$ the map $f_\lambda : U'_\lambda \rightarrow U_\lambda$ is a parabolic-like map with *the same number of attracting petals in its filled Julia set*;
5. the dividing arcs move holomorphically, i.e. we have a holomorphic motion

$$\Phi : \Lambda \times \gamma_{\lambda_0} \rightarrow \mathbb{C};$$

6. the boundaries of the codomains move holomorphically and the motion defines a piecewise C^1 -diffeomorphisms with no cusps in z , i.e. we have a holomorphic motion

$$B : \Lambda \times \partial U_{\lambda_0} \rightarrow \mathbb{C}$$

which is a piecewise C^1 -diffeomorphism with no cusps in z (for every fixed λ). Moreover, $B_\lambda(\gamma_{\lambda_0}(\pm 1)) = \gamma_\lambda(\pm 1)$.

Note that the fact that $\Phi : \Lambda \times \gamma_{\lambda_0} \rightarrow \gamma_\lambda$ is a holomorphic motion implies that the map Φ extends to a quasiconformal homeomorphism whose restriction $\Phi_\lambda : \gamma_{\lambda_0} \rightarrow \gamma_\lambda$ conjugates dynamics.

Notation. As in [LL], we will use through out this chapter both the notations

- $\gamma_\lambda : [-1, 1] \rightarrow \overline{U_\lambda}, \gamma_\lambda(0) = z_\lambda,$
- $\gamma_{\lambda+} : [0, 1] \rightarrow \overline{U_\lambda}, \gamma_{\lambda-} : [0, -1] \rightarrow \overline{U_\lambda}, \gamma_{\lambda\pm}(0) = z_\lambda, \gamma_\lambda := \gamma_{\lambda+} \cup \gamma_{\lambda-}.$

1.0.1 Remarks about the definition

Note that we require all the maps in an analytic family of parabolic-like maps to have the same number of attracting petals in its filled Julia set (see 1.1 (4)). This condition is necessary to allow us to ask a *holomorphic motion of the dividing arcs* (see 1.1 (5)). Indeed, the dividing arcs for a parabolic-like map $f_{\hat{\lambda}}$ with no attracting petals in $K_{f_{\hat{\lambda}}}$ form a cusp at the parabolic fixed point. On the other hand, the dividing arcs for a parabolic-like map $f_{\hat{\lambda}}$ with a positive number of petals in $K_{f_{\hat{\lambda}}}$ form a positive angle on both the side of $K_{f_{\hat{\lambda}}}$ and the side of $\Delta_{f_{\hat{\lambda}}}$, and it is well known that there is no quasiconformal map mapping a cusp to a curve with positive angle.

1.0.2 Degree, Filled Julia set, Julia set and connectedness locus for analytic families of parabolic-like maps.

The degree of the analytic family f_λ is independent of λ . Indeed, since the family f_λ depends holomorphically on λ , the degree depends continuously on the parameter, and since it is a natural number, it is constant, and therefore it is independent of λ . We call it the degree of \mathbf{f} .

For all $\lambda \in \Lambda$ let us call z_λ the parabolic-fixed point of f_λ , and let us set

- $K_\lambda = K_{f_\lambda},$
- $J_\lambda = J_{f_\lambda}$
- $\mathbf{K}_f = \{(\lambda, z) \mid z \in K_\lambda\}.$

The set \mathbf{K}_f is closed in $\overline{\Omega'_f}$, and since the projection from the closure of Ω'_f in \mathbf{U} to Λ is proper, the projection of \mathbf{K}_f into Λ is proper.

Define

$$M_f = \{\lambda \mid K_\lambda \text{ is connected}\}.$$

1.1 Analytic families of parabolic-like maps of degree 2

By the Straightening theorem in [LL] if f is a parabolic-like map of degree $d = 2$, f is hybrid equivalent to a member of the family

$$Per_1(1) = \{[P_A] \mid P_A(z) = z + 1/z + A, A \in \mathbb{C}\},$$

and if K_f is connected this member is unique (up to holomorphic conjugacy).

Note that, if P_{A_1} and P_{A_2} are holomorphically conjugate, then $(A_1)^2 = (A_2)^2$. Indeed, a Möbius transformation which conjugates P_{A_1} and P_{A_2} fixes

the parabolic fixed point $z = \infty$ and its preimage $z = 0$, and it can fix or interchange the critical points $z = 1$ and $z = -1$. Hence a class $[P_A]$ in $Per_1(1)$ contains two maps, i.e.

$$[P_A] = \{P_A, P_{-A}\}.$$

In the following we will refer to a quadratic rational map of the family $Per_1(1)$ as one of these representatives of its class.

The family $Per_1(1)$ is typically parametrized by $B = 1 - A^2$, which is the multiplier of the 'free' fixed point $z = -1/A$ of P_A . The connectedness locus of $Per_1(1)$ is called M_1 .

Hence if $\mathbf{f} = (f_\lambda : U'_\lambda \rightarrow U_\lambda)$ is an analytic family of parabolic-like maps of degree 2, we can define a map

$$\begin{aligned} \chi : M_f &\rightarrow M_1 \\ \lambda &\rightarrow B, \end{aligned}$$

which associates to each λ the multiplier of the fixed point of the member $[P_A]$ hybrid equivalent to f_λ .

Note that for every $A \neq 0$, the map P_A has a parabolic fixed point of parabolic multiplicity 1, while the map $P_0 = z + 1/z$ has a parabolic fixed point of parabolic multiplicity 2. Therefore, for every $A \neq 0$, a parabolic-like restriction of the map P_A has no attracting petals in its filled Julia set (for the definition of filled Julia set for a rational map and for the construction of a parabolic-like restriction of a map P_A see [LL]), while a parabolic-like restriction of P_0 has exactly one attracting petal in its filled Julia set. On the other hand, all the maps of an analytic family of parabolic-like maps have the same number of attracting petals in their filled Julia set. Each (maximal) attracting petal requires a critical point in its boundary. Hence, there are exactly 2 possibilities for the number of attracting petals in the filled Julia set of an analytic family f_λ of parabolic-like maps of degree 2. Either for each $\lambda \in \Lambda$ the map f_λ has no attracting petals in K_{f_λ} , or for each $\lambda \in \Lambda$ the map f_λ has a exactly one attracting petal in K_{f_λ} .

In the second case, all the members of \mathbf{f} are hybrid conjugate to the map $P_0 = z + 1/z$, hence the map

$$\chi : M_f \rightarrow M_1$$

is the constant map

$$\lambda \rightarrow 1,$$

(but this case is not really interesting).

On the other hand, in the first case, all the members of \mathbf{f} have no petals in their filled Julia set. This means that there is no $\lambda \in \Lambda$ such that f_λ is hybrid conjugate to the map $P_0 = z + 1/z$, and finally the range of the map χ is not the whole of M_1 , but it belongs to $M_1 \setminus \{1\}$. This is the case we are interested in.

The aim of this chapter is indeed to prove that the map χ extends to a map defined on Λ , whose restriction to M_f , under suitable conditions (see Definition 4.7) is a ramified covering of $M_1 \setminus \{1\}$. Hence in the remainder of this chapter we will consider analytic families of parabolic-like maps of degree 2.

1.2 Persistently and non persistently indifferent periodic points

Let $(R_\lambda)_{\lambda \in \Lambda}$ be an analytic family of rational maps. In the paper 'On the dynamics of rational maps' (see [MSS]), Mañé, Sad and Sullivan introduce two partitions of Λ into a dense open set of parameters, for which the family is structurally stable, and its complement. In the first partition, structural stability is required on a neighborhood of the Julia set; in the second partition it is required on the Riemann sphere. In this section we study the first partition in our setting, since parabolic-like maps is a local concept. We will see that on the structurally stable set we can construct a holomorphic motion of the Julia set, and that the structurally stable set coincides with $\Lambda \setminus \partial M_f$.

Let $\mathbf{f} = (f_\lambda : U'_\lambda \rightarrow U_\lambda)$ be an analytic family of parabolic-like mappings. An indifferent periodic point z' for f_{λ_0} , is called *persistent* if for each neighborhood $V(z')$ of z' there exists a neighborhood $W(\lambda_0)$ of λ_0 such that, for every $\lambda \in W(\lambda_0)$ the map f_λ has in $V(z')$ an indifferent periodic point z'_λ of the same period and multiplier. Hence, if for some $\hat{\lambda} \in \Lambda$ all the periodic points of $f_{\hat{\lambda}}$ are hyperbolic, then, for all $\lambda \in \Lambda$ (since Λ is connected), f_λ does not have persistently indifferent periodic points (cfr. [MSS]).

Let us define

- $I = \{\lambda \mid f_\lambda \text{ has in } \Omega'_\lambda \text{ a non persistently indifferent periodic point}\},$
- $F = \overline{I},$
- $R = \Lambda \setminus F.$

Note that:

1. for all $\lambda \in \Lambda$ the parabolic fixed point z_λ belongs to $\partial \Omega'_\lambda$ (and not to Ω'_λ);
2. the parabolic fixed point is persistent. Indeed, if $\mathbf{f} = (f_\lambda : U'_\lambda \rightarrow U_\lambda)$ is an analytic family of parabolic-like mappings and z_0 is the parabolic fixed point of f_{λ_0} , for each neighborhood $V(z_0)$ of z_0 there exists a neighborhood $W(\lambda_0)$ of λ_0 such that, for every $\lambda \in W(\lambda_0)$ the map f_λ has in $V(z_0)$ a parabolic fixed point of multiplier 1, by definition of analytic family of parabolic-like mappings.

Proposition 1.2. *Locally on R there exists a dynamic holomorphic motion of the Julia set, i.e. choosing $\lambda_0 \in R$ there exists a neighborhood $W(\lambda_0)$ of λ_0 and a map $\tau : W(\lambda_0) \times J_{\lambda_0} \rightarrow \mathbb{C}$ such that:*

1. $\forall z \in J_{\lambda_0}, \tau_{\lambda_0}(z) = z;$
2. τ is holomorphic in λ and injective in $z;$
3. $\forall \lambda \in W(\lambda_0), f_\lambda \circ \tau_\lambda = \tau_\lambda \circ f_{\lambda_0}.$

Moreover, for all $\lambda \in W(\lambda_0)$ the map $\tau_\lambda : J_{\lambda_0} \rightarrow \mathbb{C}$ is quasiconformal.

Proof. The proof follows the one in [MSS], we give it here for completeness.

Let $\lambda_0 \in R$, and let $W(\lambda_0)$ be a neighborhood of λ_0 isomorphic to a disk.

Claim 1.1. *For every repelling periodic point z_{λ_0} of f_{λ_0} there exists an analytic map*

$$\begin{aligned} z &: W(\lambda_0) \rightarrow \overline{\mathbb{C}} \\ \lambda &\rightarrow z(\lambda), \end{aligned}$$

such that $z(\lambda)$ is a repelling periodic of f_λ of the same period as z_{λ_0} .

Proof. Let z_{λ_0} be a repelling periodic point for f_{λ_0} of period k . Hence it is a solution of the equation $\psi(\lambda_0, z) = f_{\lambda_0}^k - z = 0$, and since it is a repelling point, $\partial_z \psi(\lambda_0, z_{\lambda_0}) \neq 0$. Thus by the implicit function theorem there exists $W \times V(z_0)$ neighborhood of $(\lambda_0, z_{\lambda_0})$ such that $\forall \lambda \in W \exists! z_\lambda \in V(z_0) : f_\lambda^k(z_\lambda) = z_\lambda$, i.e., there exists a holomorphic function $z(\lambda) : W \rightarrow V(z_0)$ which associates to any λ the z_λ such that $f_\lambda^k(z_\lambda) = z_\lambda$. Let $\hat{\lambda} \in \partial W \cap \overline{W}(\lambda_0)$. Then $\lim_{\lambda \rightarrow \hat{\lambda}} z(\lambda) = z(\hat{\lambda})$ is a repelling periodic point of $f_{\hat{\lambda}}$ of period k , since $W(\lambda_0) \subset R$. Then by the implicit function theorem there exists $\hat{W} \times V(z(\hat{\lambda}))$ neighborhood of $(\hat{\lambda}, z(\hat{\lambda}))$ such that $\forall \lambda \in \hat{W} \exists! \hat{z}_\lambda \in V(z(\hat{\lambda})) : f_\lambda^k(\hat{z}_\lambda) = \hat{z}_\lambda$. By uniqueness, $\forall \lambda \in W \cap \hat{W}$, $z_\lambda = \hat{z}_\lambda$, hence we can extend z_λ to all of $W(\lambda_0)$. \square

Call B_{λ_0} the set of the repelling periodic points of f_{λ_0} . Hence we obtain a holomorphic motion $\tau : W(\lambda_0) \times B_{\lambda_0} \rightarrow \mathbb{C}$ of the repelling periodic points of f_{λ_0} . Indeed:

1. $\tau_{\lambda_0} = z_0$, i.e. τ_{λ_0} is the identity on z_0 ,
2. $\forall \lambda \in W(\lambda_0)$ the map $\tau_\lambda(z_0)$ is injective,
3. the map $\tau_z(\lambda) = z_\lambda$ is holomorphic by construction.

Remark 1.1. *The condition $\forall \lambda \in W(\lambda_0)$ the map τ_λ is injective is trivially satisfied because $\lambda \in R$. Indeed, injectivity means that if there exists λ such that $\tau_\lambda(z_1) = \tau_\lambda(z_2)$, then $z_1 = z_2$. In other words, this means that the orbit $\tau_\lambda(z_1) = \tau(\lambda, z_1)$ will never cross the orbit $\tau_\lambda(z_2) = \tau(\lambda, z_2)$, when $z_1 \neq z_2$. The only case in which they can intersect is when two orbits $\tau_\lambda(z_1)$ and $\tau_\lambda(z_2)$ collapse in the same, i.e. when two hyperbolic periodic points collapse in the same parabolic one. Since we are in R , this cannot happen.*

Since the Julia set is the closure of repelling points, by the λ -Lemma we obtain a holomorphic motion of the Julia set

$$\tau : W(\lambda_0) \times J_{\lambda_0} \rightarrow \mathbb{C}.$$

This holomorphic motion is dynamic. Indeed, if z_0 is a repelling periodic point of period k for f_{λ_0} , by construction $z_\lambda = \tau_\lambda(z_0)$ is a repelling periodic point of period k for f_λ . Hence $\tau(\lambda, z)$ is a conjugacy between repelling periodic points and therefore by continuity it is a conjugacy between Julia sets. \square

Proposition 1.3. *The dynamic holomorphic motion $\tau : W(\lambda_0) \times J_{\lambda_0} \rightarrow \mathbb{C}$ constructed locally on R in Prop. 1.2 extends to a dynamic holomorphic motion*

$$\tau : W(\lambda_0) \times U(J_{\lambda_0}) \rightarrow \mathbb{C}$$

where $U(J_{\lambda_0})$ is a neighborhood of the Julia set J_{λ_0} .

For a proof we refer to [MSS] pg.210 – 215 (in the case \mathbf{f} has Siegel disks or Herman rings see the proof in [S]).

Corollary 1.1. *Let W be a connected component of R . If $\lambda_1, \lambda_2 \in W$, then $K_{\lambda_1}, K_{\lambda_2}$ are quasiconformally homeomorphic. In particular, either $W \subset M_f$ or $W \cap M_f = \emptyset$.*

Proof. If $\lambda_1, \lambda_2 \in W$, where W is a connected component of R , then J_{λ_1} and J_{λ_2} are quasiconformally homeomorphic (since there is a local holomorphic motion of the Julia set). If K_{λ_1} and K_{λ_2} have interior, let K_i, K_j be the connected components of K_{λ_1} and K_{λ_2} respectively (K_{λ_1} and K_{λ_2} have the same number of connected components, since the dynamics on J_{λ_1} and J_{λ_2} are quasiconformally conjugate). Let G_i, G_j be quasicircles in $U(J_{\lambda_1}) \cap K_{\lambda_1}^\circ$ and $U(J_{\lambda_2}) \cap K_{\lambda_2}^\circ$ respectively. Let $\phi_i : \hat{K}_i \rightarrow \mathbb{D}$, $\phi_j : \hat{K}_j \rightarrow \mathbb{D}$ be Riemann maps and define $\mathbb{S}_i = \phi(G_i)$ and $\mathbb{S}_j = \phi(G_j)$. Then the homeomorphism $\varphi := \phi_j \circ \tau \circ \phi_i^{-1} : \mathbb{S}_i \rightarrow \mathbb{S}_j$ is quasimetric, hence it extends to a quasiconformal map $\Phi : \mathbb{D}_i \rightarrow \mathbb{D}_j$. Therefore we can define a quasiconformal homeomorphism $\phi_j^{-1} \circ \Phi \circ \phi_i : K_i \rightarrow K_j$ between every connected component of K_{λ_1} and K_{λ_2} respectively, and thus K_{λ_1} is quasiconformally homeomorphic to K_{λ_2} .

Finally, either $\lambda_1, \lambda_2 \in M_f$, or both $\lambda_1, \lambda_2 \notin M_f$, since there can not be a homeomorphism between a connected set and a disconnected one. \square

Proposition 1.4. (a) *The interior of $M_f \subset R$*

$$(b) R = \Lambda \setminus \partial M_f$$

Proof. The proof follows the one in [DH]. We give it here for completeness.

(a) Choose $\lambda_0 \in \mathring{M}_f$, and suppose f_{λ_0} has a non-persistent indifferent periodic point α_0 of period k and multiplicity n . Let $V(\alpha_0)$ be a round disk neighborhood of α_0 such that α_0 is the only periodic point of period k in $\overline{V}(\alpha_0)$. Let Λ_0 be a neighborhood of λ_0 in \mathring{M}_f such that for all $\lambda \in \Lambda_0$ f_λ has in $V(\alpha_0)$ n periodic points counted with multiplicity and λ_0 is the only parameter in Λ_0 such that f_{λ_0} has in $V(\alpha_0)$ a degenerate periodic point of period k . Let $W(0)$ be a n -covering of Λ_0 branched at 0. Then there exists a branched covering $\lambda : W(0) \rightarrow \Lambda_0$, $t \rightarrow t^n + \lambda_0$, such that $\lambda(0) = \lambda_0$. Note that if α_0 is a simple indifferent periodic point, the map λ is the translations by λ_0 .

Hence $\forall t \in W(0) \exists \alpha_{\lambda(t)} \in V(\alpha_0) : f_{\lambda(t)}^k(\alpha_{\lambda(t)}) = \alpha_{\lambda(t)}$, i.e., there exists a holomorphic map

$$\begin{aligned} \alpha : W(0) &\rightarrow \mathbb{C} \\ t &\rightarrow \alpha(t), \end{aligned}$$

such that $\alpha(0) = \alpha_0$ and for all t we have that $\alpha(t)$ is a periodic point of period k and multiplier $\rho(t)$ for $f_{\lambda(t)}$, where

$$\rho : W(0) \rightarrow \mathbb{C}^*$$

is a non constant holomorphic function (holomorphic because the $f_\lambda(z)$ is holomorphic in both λ and z , and non constant since the indifferent periodic point α_0 is non persistent). Again by the implicit function theorem the critical point

moves holomorphically, i.e. there exists a holomorphic map $\omega : W(0) \rightarrow \mathbb{C}$, $t \rightarrow \omega(t)$, such that $(f_{\lambda(t)})'(\omega(t)) = 0$.

Let (t_n) be a sequence in $W(0)$ converging to 0, such that $|\rho(t_n)| < 1 \forall n$. Then, for each n , $\alpha(t_n)$ is an attracting periodic point of period k for $f_{\lambda(t_n)}$. Hence the critical point belongs to the attracting basin of $\alpha(t_n)$ (and $\exists i$, $0 \leq i \leq k$, : $f_{\lambda(t_n)}^i(\omega(t_n))$ belongs to the immediate basin of attraction of $\alpha(t_n)$). Therefore, for each n , we have:

$$f_{\lambda(t_n)}^{i+kp}(\omega(t_n)) \rightarrow \alpha(t_n) \text{ as } p \rightarrow \infty.$$

Let us define the sequence

$$\begin{aligned} F_p &: W(0) \rightarrow \mathbb{C} \\ t &\rightarrow f_{\lambda(t)}^{i+kp}(\omega(t)). \end{aligned}$$

Thus $\{F_p\}_{p \in \mathbb{N}}$ is a family of analytic maps (since f_λ are analytic) bounded on any compact subset of $W(0)$ (indeed since $\lambda(t) \in M_f$ for every t , then $F_p(t) \in K_{\lambda(t)}$). In particular it is a normal family. Suppose F_{p_n} is a subsequence converging to some function $h : W(0) \rightarrow \mathbb{C}$ as $t_n \rightarrow 0$. Then $h(t_n) = \alpha(t_n)$ for all n . Hence $h = \alpha$ and $F_p \rightarrow \alpha$, by the uniqueness of analytic continuation. But in $W(0)$ there are points t^* such that $|\rho(t^*)| > 1$, then $\alpha(t^*)$ is a repelling periodic point, hence it cannot attract the sequence $F_p(t)$. Thus $\dot{M}_f \cap I = \emptyset$, and since \dot{M}_f is open, $\dot{M}_f \cap F = \emptyset$ and finally $\dot{M}_f \subset R$.

(b) For the previous corollary, if W is a connected component of R , then $W \subset M_f$ or $W \cap M_f = \emptyset$. This implies that $R \cap \partial M_f = \emptyset$. Therefore $R \subset \Lambda \setminus \partial M_f$.

By (a) $\dot{M}_f \subset R$, then we need to prove $(\Lambda \setminus M_f) \subset R$.

For any $\lambda \in \Lambda$, since $d = 2$ the map f_λ has a unique critical point ω_λ . If $\lambda \in (\Lambda \setminus M_f)$ then $\omega_\lambda \notin K_\lambda$. Hence $\omega_\lambda \in (U'_\lambda \setminus K_\lambda)$, and any periodic point of f_λ which is not the parabolic fixed point is repelling. Therefore $(\Lambda \setminus M_f) \cap I = \emptyset$, and since $\Lambda \setminus M_f$ is open, $(\Lambda \setminus M_f) \subset R$. □

2 Holomorphic motion of a fundamental annulus A_{λ_0} and Tubings

In [LL] we proved that a degree 2 parabolic-like map is hybrid conjugate to a member of the family $Per_1(1)$, by changing its external class into h_2 , the external class of the family $Per_1(1)$. In other words we glued outside a degree 2 parabolic-like map f the map h_2 . More precisely, we constructed a quasiconformal C^1 diffeomorphism $\tilde{\psi}$ between a *fundamental annulus* of the parabolic-like map and a *fundamental annulus* of h_2 . Then we pulled back by $\tilde{\psi}$ the standard structure σ_0 , and we obtained an almost complex structure σ_1 . In order to spread σ_1 by the dynamics, we replaced the parabolic-like map with h_2 on Δ , and hence we obtain a bounded almost complex structure σ invariant under the dynamics of the new map \tilde{f} . Finally, by integrating σ we could construct a parabolic-like map hybrid conjugate to f and with external map h_2 .

In this chapter we want to perform this surgery for an analytic family of parabolic-like maps, and we want to do it with some regularity with respect

to the parameter. Hence we start by defining a family of quasiconformal maps between a fundamental annulus of h_2 and a fundamental annulus of f_λ which depends holomorphically on the parameter. In analogy with the polynomial-like setting we will call this family a *holomorphic Tubing*. Therefore we should start by constructing a *fundamental annulus* for h_2 and for $(f_\lambda)_{\lambda \in \Lambda}$.

In [LL] we already constructed a quasiconformal C^1 -diffeomorphism $\tilde{\psi}$ between a fundamental annulus of the parabolic-like map and a fundamental annulus of h_2 . That construction shows that the fundamental annulus for h_2 depends on the parabolic-like map we started with. Therefore in this section we will first fix a $\lambda_0 \in \Lambda$, construct a fundamental annulus for h_2 and one for f_{λ_0} , and recall the quasiconformal C^1 -diffeomorphism $\tilde{\psi}$ between these fundamental annuli. Then we will derive fundamental annuli for f_λ from the fundamental annulus of f_{λ_0} , and we will construct a *holomorphic motion* between them. Finally we will obtain a *holomorphic Tubing* by composing $\tilde{\psi}$ and the holomorphic motion.

Notation. The term *fundamental annulus* is used here not in the sense of covering maps.

2.0.1 A fundamental annulus A for h_2

The map $h_2(z) = \frac{z^2+1/3}{1+z^2/3}$ is the external map of the family $Per_1(1)$ (cfr. [LL]). Let $h_2 : W' \rightarrow W$ (where $W = \{z : \exp(-\epsilon) < |z| < \exp(\epsilon)\}$, $\epsilon > 0$, and $W' = h_2^{-1}(W)$) be a degree 2 covering. Choose $\lambda_0 \in \Lambda$. Let h_{λ_0} be an external map of f_{λ_0} , and z_0 be its parabolic fixed point and define $\gamma_{h_{\lambda_0}+} = \alpha_{\lambda_0}(\gamma_{\lambda_0+})$, $\gamma_{h_{\lambda_0}-} = \alpha_{\lambda_0}(\gamma_{\lambda_0-})$ (where $\alpha : \mathbb{C} \setminus K_\lambda \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ is the isomorphism which defines h_{λ_0}). Let $\Xi_{h_f \pm}$ be repelling petals for the parabolic fixed point z_0 which intersect the unit circle and $\phi_\pm : \Xi_{h_f \pm} \rightarrow \mathbb{H}_-$ be Fatou coordinates for h_{λ_0} with axis tangent to the unit circle at the parabolic fixed point z_0 .

Let $\Xi_{h_2 \pm}$ be repelling petals which intersect the unit circle for the parabolic fixed point $z = 1$ of h_2 , and let $\tilde{\phi}_\pm : \Xi_{h_2 \pm} \rightarrow \mathbb{H}_-$ be Fatou coordinates for h_2 with axis tangent to the unit circle at 1.

Define $\tilde{\gamma}_+ = \tilde{\phi}_+^{-1}(\phi_{h_{\lambda_0}+}(\gamma_{h_{\lambda_0}+}))$ and $\tilde{\gamma}_- = \tilde{\phi}_-^{-1}(\phi_{h_{\lambda_0}-}(\gamma_{h_{\lambda_0}-}))$.

Define $\tilde{\Delta}_W = h_2(\Delta_W \cap \Delta'_W)$, $\tilde{W} = \Omega_W \cup \tilde{\gamma} \cup \tilde{\Delta}_W$, $\tilde{W}' = h_2^{-1}(\tilde{W})$, $\tilde{\Omega}'_W = \Omega'_W \cap \tilde{W}'$, $\tilde{\Delta}'_W = \Delta'_W \cap \tilde{W}'$ and $Q_W = \Omega_W \setminus \tilde{\Omega}'_W$. We call *fundamental annulus* for h_2 the topological annulus $A = W \setminus (\tilde{\Omega}'_W \cup \mathbb{D})$.

2.0.2 A fundamental annulus A_{λ_0} for f_{λ_0} and a quasiconformal C^1 diffeomorphism $\tilde{\Psi} : A \rightarrow A_{\lambda_0}$

Let ψ be a quasiconformal map between ∂U_{λ_0} and the outer boundary of W , such that $\psi(\gamma_{\lambda_0+}(1)) = \tilde{\gamma}_+(1)$ and $\psi(\gamma_{\lambda_0-}(1)) = \tilde{\gamma}_-(1)$. Let $\Phi_{\Delta_{\lambda_0}} : \Delta_{\lambda_0} \rightarrow \Delta_W$ be a quasiconformal C^1 diffeomorphism which extends by ψ on ∂U_{λ_0} and by $\tilde{\phi}_\pm^{-1} \circ \phi_{h_{\lambda_0} \pm} \circ \alpha_{\lambda_0}$ on $\gamma_{\lambda_0 \pm}$ (cfr. the proof of Prop. 2.7 in [LL]). Define $\tilde{\Delta}_{\lambda_0} = \Phi_{\Delta_{\lambda_0}}^{-1}(\tilde{\Delta}_W)$, $\tilde{\Delta}'_{\lambda_0} = \Phi_{\Delta_{\lambda_0}}^{-1}(\tilde{\Delta}'_W)$, $\tilde{U}_{\lambda_0} = (\Omega_{\lambda_0} \cup \gamma_{\lambda_0} \cup \tilde{\Delta}_{\lambda_0}) \subset U_{\lambda_0}$.

Consider

$$\tilde{f}_{\lambda_0}(z) = \begin{cases} \Phi_{\Delta_{\lambda_0}}^{-1} \circ h_2 \circ \Phi_{\Delta_{\lambda_0}} & \text{on } \tilde{\Delta}'_{\lambda_0} \\ f_{\lambda_0} & \text{on } \Omega'_{\lambda_0} \cup \gamma_{\lambda_0} \end{cases}$$

Define $\widetilde{U}'_{\lambda_0} = \widetilde{f}_{\lambda_0}^{-1}(\widetilde{U}_{\lambda_0})$, $Q_{\lambda_0} = \Omega_{\lambda_0} \setminus \overline{\Omega'}_{\lambda_0}$, and the *fundamental annulus* $A_{\lambda_0} = U_{\lambda_0} \setminus \overline{\Omega'}_{\lambda_0}$. Let $\bar{\psi} : \partial\widetilde{U}_{\lambda_0} \rightarrow \partial(\widetilde{W} \cup \mathbb{D})$ be quasiconformal map coinciding with ψ on the outer boundary of Ω_{λ_0} , and let $\psi_1 : \partial\widetilde{U}'_{\lambda_0} \rightarrow \partial(\widetilde{W}' \cup \mathbb{D})$ be the lift of $\bar{\psi} \circ \widetilde{f}_{\lambda_0}$ to h_2 which preserves the dynamics on the dividing arcs.

Let $\Phi_{Q_{\lambda_0}} : Q_{\lambda_0} \rightarrow Q_W$ be a quasiconformal C^1 diffeomorphism which extends by $\bar{\psi}$ on $\partial\Omega_{\lambda_0}$, by ψ_1 on $\partial\widetilde{\Omega}_{\lambda_0}$ and by $\bar{\phi}_{\pm}^{-1} \circ \phi_{h_{\lambda_0} \pm} \circ \alpha_{\lambda_0}$ on $\gamma_{\lambda_0 \pm}$ (cfr. the proof of Prop 2.7 in [LL]). Define a map $\tilde{\psi} : A_{\lambda_0} \rightarrow A$ as follows :

$$\tilde{\psi}(z) = \begin{cases} \bar{\phi}_{\pm}^{-1} \circ \phi_{h_{\lambda_0} \pm} \circ \alpha_{\lambda_0} & \text{on } \gamma_{\lambda_0 \pm} \\ \Phi_{\Delta_{\lambda_0}} & \text{on } \Delta_{\lambda_0} \\ \Phi_{Q_{\lambda_0}} & \text{on } Q_{\lambda_0} \end{cases}$$

This map is a quasiconformal C^1 diffeomorphism (see the proof of Prop 2.7 in [LL]) and it extends quasiconformally to the boundaries. Therefore the map $\tilde{\Psi} := \tilde{\psi}^{-1} : A \rightarrow A_{\lambda_0}$ is a quasiconformal C^1 diffeomorphism which extends quasiconformally to the boundaries.

2.0.3 Holomorphic motion of the fundamental annulus A_{λ_0}

Define for all $\lambda \in \Lambda$ the set $a_{\lambda} = U_{\lambda} \setminus \overline{\Omega'}_{\lambda}$. Then the set a_{λ} is a topological annulus. Define the map $\tilde{\tau} : \Lambda \times \partial a_{\lambda_0} \rightarrow \partial a_{\lambda}$ as follows:

$$\tilde{\tau}(z) = \begin{cases} \Phi_{\lambda} & \text{on } \gamma_{\lambda_0} \\ B_{\lambda} & \text{on } \partial U_{\lambda_0} \\ f_{\lambda}^{-1} \circ B_{\lambda} \circ f_{\lambda_0} & \text{on } \partial U'_{\lambda_0} \cap \partial \Omega'_{\lambda_0} \end{cases}$$

Let us show that $\tilde{\tau}$ is a holomorphic motion with basepoint λ_0 .

Indeed:

1. $\forall z_0 \in \gamma_{\lambda_0}$, $\tilde{\tau}(z_0) = \Phi_{\lambda_0}(z_0) = z_0$ since Φ is a holomorphic motion, $\forall z_0 \in \partial U_{\lambda_0}$, $\tilde{\tau}(z_0) = Id(z_0) = z_0$, and $\forall z_0 \in \partial U'_{\lambda_0} \cap \partial \Omega'_{\lambda_0}$, $\tilde{\tau}(z_0) = f_{\lambda_0}^{-1} \circ Id \circ f_{\lambda_0} = z_0$;
2. the map $\tilde{\tau}$ is injective in z , since Φ_{λ} and B_{λ} are holomorphic motions with disjoint images on $\partial a_{\lambda_0} \setminus \gamma_{\lambda_0 \pm}(\pm 1)$, and $f_{\lambda} : \partial U'_{\lambda} \rightarrow \partial U_{\lambda}$ is a degree d covering;
3. the map $\tilde{\tau}$ is holomorphic in λ , since Φ_{λ} and B_{λ} are holomorphic motions, and the map f_{λ} depends holomorphically on λ .

Since $\Lambda \approx \mathbb{D}$, by the Slodkowski's theorem we can extend $\tilde{\tau}$ to a holomorphic motion $\tilde{\tau} : \Lambda \times \mathbb{C} \rightarrow \mathbb{C}$. In particular we obtain a holomorphic motion of the set $\Lambda \times \widetilde{U}_{\lambda_0}$. For every $\lambda \in \Lambda$ define $\widetilde{U}_{\lambda} = \tilde{\tau}(\widetilde{U}_{\lambda_0})$, and $\widetilde{\Delta}'_{\lambda} = \tilde{\tau}(\widetilde{\Delta}'_{\lambda_0})$. Define for every $\lambda \in \Lambda$ the map \tilde{f}_{λ} as follows:

$$\tilde{f}_{\lambda}(z) = \begin{cases} \tilde{\tau} \circ \tilde{\Psi} \circ h_2 \circ \tilde{\Psi}^{-1} \circ \tilde{\tau}^{-1} & \text{on } \widetilde{\Delta}'_{\lambda} \\ f_{\lambda} & \text{on } \Omega'_{\lambda} \cup \gamma_{f_{\lambda}} \end{cases}$$

and the set $\widetilde{U}'_{\lambda} = \tilde{f}_{\lambda}^{-1}(\widetilde{U}_{\lambda})$. Finally, define for all $\lambda \in \Lambda$ the set $A_{\lambda} = U_{\lambda} \setminus \overline{\Omega'}_{\lambda}$. Then the set A_{λ} is a topological annulus, and we call it the *fundamental annulus*

of f_λ . The holomorphic motion $\tilde{\tau} : \Lambda \times \widehat{\mathbb{C}} \rightarrow \mathbb{C}$ restricts to a holomorphic motion

$$\widehat{\tau} : \Lambda \times A_{\lambda_0} \rightarrow A_\lambda$$

which respects the dynamics. Note that, by construction, this holomorphic motion extends to the boundaries, and the extension respects the dynamics.

2.0.4 Holomorphic Tubings

Define $T := \widehat{\tau} \circ \tilde{\Psi} : \Lambda \times A \rightarrow A_\lambda$. The map T is not a holomorphic motion, since $T_{\lambda_0} = \tilde{\Psi} \neq Id$, but nevertheless it is quasiconformal in z for every fixed $\lambda \in \Lambda$ and holomorphic in λ for every fixed $z \in A$.

Definition 2.1. Let us denote by **holomorphic tubing** the map $T := \widehat{\tau} \circ \tilde{\Psi} : \Lambda \times A \rightarrow A_\lambda$.

By construction $T_{\lambda_0} = \widehat{\tau} \circ \tilde{\Psi} = Id \circ \tilde{\Psi} = \tilde{\psi}^{-1} : A \rightarrow A_{\lambda_0}$ is a quasiconformal map which defines an almost complex structure μ_{λ_0} which gives a straightening map. Indeed, for every $\lambda \in \Lambda$ we define on U_λ the Beltrami form μ_λ as follows:

$$\mu_\lambda(z) = \begin{cases} \mu_{\lambda,0} = \widehat{T}_{\lambda*}(\sigma_0) & \text{on } A_\lambda \\ \mu_{\lambda,n} = (\tilde{f}_\lambda^n)^* \mu_{\lambda,0} & \text{on } A_{\lambda,n} = (\tilde{f}_\lambda)^{-n}(Q_{f_\lambda} \cup \tilde{\Delta}_\lambda) \\ 0 & \text{on } K_\lambda \end{cases}$$

For every λ the map \widehat{T}_λ is quasiconformal, then its inverse is quasiconformal, hence $\|\mu_{\lambda,0}\|_\infty \leq k < 1$ on every compact subset of Λ . On $\tilde{\Omega}'_\lambda$ the Beltrami form $\mu_{\lambda,n}$ is obtained by spreading $\mu_{\lambda,0}$ by the dynamics of f_λ , which is holomorphic, while on Δ_λ the Beltrami form $\mu_{\lambda,n}$ is constant for all n (indeed on Δ_λ the Beltrami form is defined by $\mu_{\lambda,0} = \widehat{T}_{\lambda*}(\sigma_0)$, and $\mu_{\lambda,n} = \mu_{\lambda,0}$, $\forall n$). Hence the dilatation of $\mu_{\lambda,i}$ is constant. Therefore $\|\mu_\lambda\|_\infty = \|\mu_{\lambda,0}\|_\infty = \sup_z |\mu_\lambda(z)|$ which is bounded. By the measurable Riemann mapping theorem ([Ah]) for every $\lambda \in \Lambda$ there exists $\phi_\lambda : U_\lambda \rightarrow \mathbb{D}$ quasiconformal map such that $(\phi_\lambda)^* \mu_0 = \mu_\lambda$. Finally, for every $\lambda \in \Lambda$ the map $g_\lambda = \phi_\lambda \circ f_\lambda \circ \phi_\lambda^{-1}$ is the parabolic-like map hybrid conjugate to f_λ and holomorphically conjugate to P_A .

Remark 2.1. By construction, the holomorphic motion $\widehat{\tau} : \Lambda \times A_{\lambda_0} \rightarrow A_\lambda$ extends to a holomorphic motion of the boundaries, and the quasiconformal C^1 diffeomorphism $\tilde{\Psi} : A \rightarrow A_{\lambda_0}$ extends as a quasiconformal map between closed annuli. Therefore, a holomorphic Tubing extends to a holomorphic tubing $T : \Lambda \times \bar{A} \rightarrow \bar{A}_\lambda$ between closed annuli.

2.0.5 Lifting Tubings

By construction, a holomorphic Tubing $T : A \rightarrow A_\lambda$ extends to a holomorphic tubing $T : \Lambda \times \bar{A} \rightarrow \bar{A}_\lambda$ from a closed fundamental annulus of h_2 to a closed fundamental annulus of f_λ (and the extension respects the dynamics, see 2.0.3). In the case K_λ is connected, it is possible to lift a Tubing to all of $W \setminus \overline{\mathbb{D}}$, while in the case K_λ is not connected, we can lift a Tubing untill we reach the critical value. Indeed, define $A_{\lambda,0} = \widetilde{U}_\lambda \setminus \tilde{\Omega}_\lambda$, $B_{\lambda,1} = \tilde{f}_\lambda^{-1}(A_{\lambda,0})$, $A_0 = \widetilde{W}_\lambda \setminus \tilde{\Omega}_W$ and $B_1 = h_2^{-1}(A_0)$. Hence $\tilde{f}_\lambda : B_{\lambda,1} \rightarrow A_{\lambda,0}$ and $h_2 : B_1 \rightarrow A_0$ are degree 2 covering maps, and, since by construction $T_\lambda(A_0) = A_{\lambda,0}$, we can lift the Tubing to

$T_{\lambda,1} := \tilde{f}_\lambda^{-1} \circ T_\lambda \circ h_2 : B_1 \rightarrow B_{\lambda,1}$ (where f_λ^{-1} , h_2^{-1} are the branches which preserve the dynamics on the overlapping domains).

Let us set $A_{\lambda,1} = B_{\lambda,1} \cap \tilde{U}$, $B_{\lambda,2} = \tilde{f}_\lambda^{-1}(A_{\lambda,1})$, $A_1 = B_1 \cap \tilde{W}$ and $B_2 = h_2^{-1}(A_1)$. Hence $\tilde{f}_\lambda : B_{\lambda,2} \rightarrow A_{\lambda,1}$ and $h_2 : B_2 \rightarrow A_1$ are degree 2 covering maps, and we can lift the Tubing to $T_{\lambda,2} := \tilde{f}_\lambda^{-1} \circ T_{\lambda,1} \circ h_2 : B_2 \rightarrow B_{\lambda,2}$ (such that $T_{\lambda,2} = T_{\lambda,1}$ on $B_2 \cap B_1$).

Finally, define recursively $A_{\lambda,n} = B_{\lambda,n} \cap \tilde{U}$, $B_{\lambda,n+1} = \tilde{f}_\lambda^{-1}(A_{\lambda,n})$, $A_n = B_n \cap \tilde{W}$ and $B_{n+1} = h_2^{-1}(A_n)$. Hence $\tilde{f}_\lambda : B_{\lambda,n+1} \rightarrow A_{\lambda,n}$ and $h_2 : B_{n+1} \rightarrow A_n$ are degree 2 covering maps, and we can lift the Tubing to $T_{\lambda,n+1} := \tilde{f}_\lambda^{-1} \circ T_{\lambda,n} \circ h_2 : B_{n+1} \rightarrow B_{\lambda,n+1}$ (such that $T_{\lambda,n+1} = T_{\lambda,n}$ on $B_{n+1} \cap B_n$). Note that, in the case K_λ is connected, we can lift the Tubing T_λ for all n . On the other hand, if K_λ is not connected, the maximum domain we can lift the Tubing T_λ to is B_{n_0} , such that B_{λ,n_0} contains the critical value of f_λ .

3 Properties of the map χ .

3.1 Extending the map χ to all of Λ .

By [LL] if f is a parabolic-like map of degree $d = 2$, f is hybrid equivalent to a member of the family $Per_1(1)$, and if K_f is connected this member is unique. Therefore, if $\mathbf{f} = (f_\lambda : U'_\lambda \rightarrow U_\lambda)$ is an analytic family of parabolic-like maps of degree 2, the map

$$\begin{aligned} \chi : M_f &\rightarrow M_1 \setminus \{1\} \\ \lambda &\rightarrow B, \end{aligned}$$

which associates to each $\lambda \in M_f$ the multiplier of the fixed point of the map P_A hybrid equivalent to f_λ is well defined (see 1.1).

By Tubings we can extend the map χ to the whole parameter space Λ . Since Tubings are not unique, the extension given here (which follows the one in [DH]) is not canonical, but it is anyway, given a Tubing, the 'natural' one.

Note that the range of the extension of χ to the whole of Λ is a *proper subset* of \mathbb{C} . Indeed, as we pointed out in 1.1, there is no $\lambda \in \Lambda$ such that f_λ is hybrid equivalent to $P_0 = z + 1/z$. Hence the range of the extension of the map χ to the whole of Λ does not contain the parameter $B = 1$, the so-called *root*.

Let T_λ be a holomorphic tubing for the analytic family of parabolic-like maps \mathbf{f} . Call ω_λ the critical point of f_λ and let n be such that $f_\lambda^n(\omega_\lambda) \in A_\lambda$, $f_\lambda^{n-1}(\omega_\lambda) \notin A_\lambda$. Hence we can iteratively lift the holomorphic tubing T_λ to $T_{\lambda,n-1} := \tilde{f}_\lambda^{-1} \circ T_{\lambda,n-2} \circ h_2 = f_\lambda^{-(n-1)} \circ T_\lambda \circ h_2^{n-1} : B_{n-1} \rightarrow B_{\lambda,n-1}$ (where h_2^{n-1} , $\tilde{f}_\lambda^{-(n-1)}$ are the branches which preserve the dynamics on the overlapping domains, see 2.0.5).

We can therefore extend the map χ to the whole of Λ by setting:

$$\begin{aligned} \chi : \Lambda \setminus M_f &\rightarrow \mathbb{C} \setminus M_1 \\ \lambda &\rightarrow \Phi^{-1} \circ T_{\lambda,n-1}^{-1}(\omega_\lambda) \end{aligned}$$

where $\Phi : \mathbb{C} \setminus M_1 \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ is the canonical isomorphism between the complement of M_1 and the complement of the unit disk. Since the maps $h_2 : B_{n-1} \rightarrow A_{n-2}$

and $\tilde{f}_\lambda : B_{\lambda,n-1} \rightarrow A_{\lambda,n-2}$ are degree 2 coverings, the map Φ is an isomorphism, and the Tubing T_λ is a holomorphic tubing (and then quasiconformal in z) the map $\chi : \Lambda \setminus M_f \rightarrow \mathbb{C} \setminus M_1$ is quasiregular.

3.2 Continuity of χ in degree 2.

In this section we want to prove that the map $\chi : \Lambda \rightarrow \mathbb{C}$ is continuous. Since the map $\chi : \Lambda \setminus M_f \rightarrow \mathbb{C} \setminus M_1$ is quasiregular, we will start by proving that χ is continuous on M_f , and then we will prove continuity on the whole of Λ .

For every $\lambda \in M_f$ the parabolic-like map f_λ is hybrid conjugate to a unique member of the family $Per_1(1)$. This means that, if μ, μ' are two different Beltrami forms on U_λ obtained by spreading by the dynamics of $\tilde{f}_\lambda, \tilde{f}'_\lambda$ the pull back of the standard structure under two different quasiconformal maps $\phi : A_\lambda \rightarrow A$, and $\phi' : A_\lambda \rightarrow A$, then $P_{A(\lambda)} = \phi \circ \tilde{f}_\lambda \circ \phi^{-1}$ and $P_{A'(\lambda)} = \phi' \circ \tilde{f}'_\lambda \circ \phi'^{-1}$ are in the same class $[P_A]$.

For this reason we are free to use a different Tubing $T'_\lambda = \hat{\tau}' \circ \tilde{\Psi} : A \rightarrow A_{\lambda_0}$ which defines a different almost complex structure μ'_λ on U_λ but yields to the same class hybrid conjugate to f_λ .

We will indeed define a different Tubing, since to prove continuity of the straightening map on M_f we will need the Tubing to be a C^1 -diffeomorphism in z . Therefore we start by defining a *diffeomorphic motion*, i.e. a map no longer holomorphic in λ and quasiconformal in z but a C^1 diffeomorphism in z continuous in both (λ, z) .

3.2.1 Diffeomorphic motion.

Let α_λ be a family of Riemann maps $\alpha : \mathbb{C} \setminus K_\lambda \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$, normalized by $\alpha_\lambda(\infty) = \infty$ and $\alpha_\lambda(\gamma_\lambda(t)) \rightarrow 1$ as $t \rightarrow 0$. Since we can define locally on R a holomorphic motion of the Julia set (cfr. 1.2), and $M_f \subset R$ (cfr. 1.4), the family α_λ is continuous on λ . Let $h_\lambda : W'_\lambda \rightarrow W_\lambda$ be the associated family of external maps (see [LL]), then h_λ depends holomorphically on z and continuously on λ , i.e. h_λ is a continuous family of holomorphic maps.

Define the dividing arcs $\gamma_{h_\lambda \pm} = \alpha_\lambda(\gamma_\lambda \pm)$, and note that the map α_λ extends to a homeomorphism $\alpha_\lambda : \gamma_\lambda \rightarrow \gamma_{h_\lambda}$ conjugating the dynamics of f_λ and h_λ . Define the set $A_{h_\lambda} = \alpha_\lambda(A_\lambda)$. Then the set A_{h_λ} is a topological annulus, and we call it the *fundamental annulus* for h_λ . We will construct a motion of the annulus A_{λ_0} by constructing a motion of the annulus $A_{h_{\lambda_0}}$.

The holomorphic motion $\hat{\tau} : \Lambda \times A_{\lambda_0} \rightarrow A_\lambda$ extends by the λ -Lemma (cfr. [MSS]) to a holomorphic motion of the boundaries $\hat{\tau} : \Lambda \times \partial A_{\lambda_0} \rightarrow \partial A_\lambda$. Therefore, the family $\alpha_\lambda \circ \hat{\tau} \circ \alpha_{\lambda_0}^{-1} : \Lambda \times \partial A_{h_{\lambda_0}} \rightarrow \partial A_{h_\lambda}$ is a family of homeomorphisms, (since α_λ extends to a homeomorphism conjugating the dynamics on the arcs), quasisymmetric on $A_{h_{\lambda_0}} \setminus z_0$, (where z_0 is the parabolic fixed point of h_{λ_0}) and continuous in (λ, z) (see Fig.1).

Let us show that the family $\alpha_\lambda \circ \hat{\tau} \circ \alpha_{\lambda_0}^{-1}$ is quasisymmetric on a neighborhood of the parabolic fixed point z_0 .

Let $\Xi_{h_{\lambda_0}+}, \Xi_{h_{\lambda_0}-}, \Xi_{h_\lambda+},$ and $\Xi_{h_\lambda-}$ be the repelling petals where $\gamma_{h_{\lambda_0}+}, \gamma_{h_{\lambda_0}-}, \gamma_{h_\lambda+},$ and $\gamma_{h_\lambda-}$, respectively reside, and let $\phi_{h_{\lambda_0} \pm} : \Xi_{h_{\lambda_0} \pm} \rightarrow \mathbb{H}_\pm$, and

$\phi_{h_\lambda \pm} : \Xi_{h_\lambda \pm} \rightarrow \mathbb{H}_-$ be Fatou coordinates, normalized by mapping the unit circle to the negative real axis. For every $\lambda \in M_f$ let $m_{\lambda+}, m_{\lambda-}$ be a sequence of real numbers continuous in λ , and set $\gamma_{s_\lambda+}(t) = \phi_{h_\lambda+}^{-1}(\log_d(t) - m_{\lambda+}i)$, $0 \leq t \leq 1$, $\gamma_{s_\lambda-}(t) = \phi_{h_\lambda-}^{-1}(\log_d(-t) + m_{\lambda-}i)$, $-1 \leq t \leq 0$. Define for every $\lambda \in M_f$ the translations $T_{(\lambda_0, \lambda)+} = m_{\lambda_0+}i - m_{\lambda+}i$ and $T_{(\lambda_0, \lambda)-} = -m_{\lambda_0+}i + m_{\lambda+}i$. By proposition 3.10, (3) there exist a quasimetric conjugacy $\delta_0 : \gamma_{h_{\lambda_0}} \rightarrow \gamma_{s_{\lambda_0}}$ between h_{λ_0} and itself and, for every λ , there exist quasimetric conjugacies $\delta_\lambda : \gamma_{h_\lambda} \rightarrow \gamma_{s_\lambda}$ between h_λ and itself. The proof of proposition 3.10, (1) shows that for every λ the map $\phi_{h_\lambda}^{-1} \circ T_{(\lambda_0, \lambda)} \circ \phi_{h_{\lambda_0}} : \gamma_{s_{\lambda_0}} \rightarrow \gamma_{s_\lambda}$ is a quasimetric conjugacy between h_{λ_0} and h_λ . Writing the map $\alpha_\lambda \circ \hat{\tau} \circ \alpha_{\lambda_0}^{-1} : \gamma_{h_{\lambda_0}} \rightarrow \gamma_{h_\lambda}$ as $\delta_\lambda^{-1} \circ \phi_{h_\lambda}^{-1} \circ T_{(\lambda_0, \lambda)} \circ \phi_{h_{\lambda_0}} \circ \delta_0$, is now clear that this map is quasimetric on a neighborhood of the parabolic fixed point z_0 .

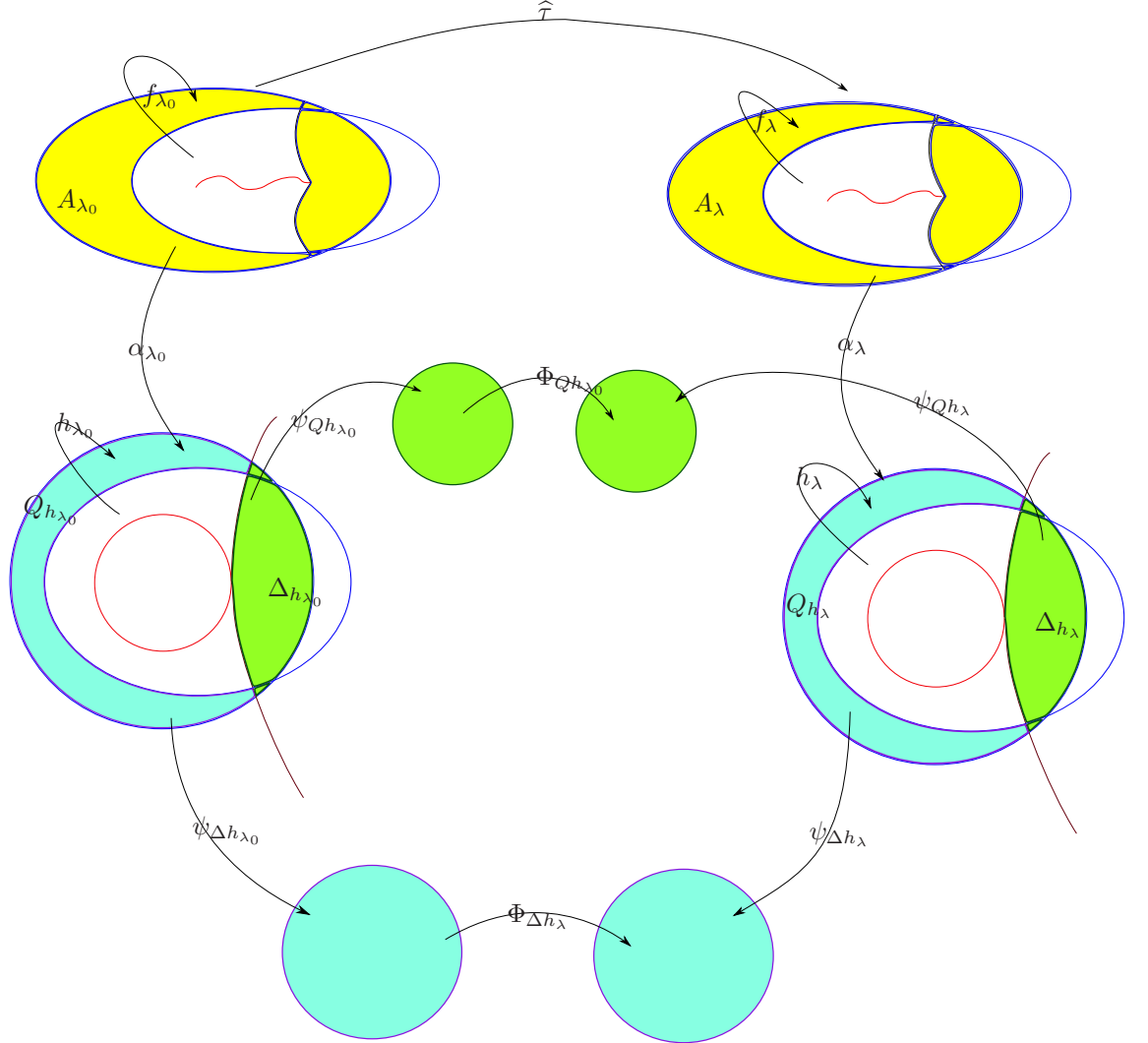


Figure 1: Construction of the diffeomorphic motion $\hat{\tau}' : A_{\lambda_0} \times M_f \rightarrow A_\lambda$.

Consider the topological annulus A_{h_λ} as the union of two quasidisks: $Q_{h_\lambda} = \Omega_{h_\lambda} \setminus \overline{\Omega'}_{h_\lambda}$ and Δ_{h_λ} (see Figure 1). The sets ∂Q_{h_λ} and $\partial \Delta_{h_\lambda}$ are quasicircles, since they are piecewise C^1 closed curves with non zero interior angles. Indeed, $\gamma_{h_\lambda+}$ and $\gamma_{h_\lambda-}$ are tangent to \mathbb{S}^1 at the parabolic fixed point (see [LL]), and we can assume the angles between $\gamma_{h_\lambda\pm}$ and $\partial(W_\lambda \cup \mathbb{D})$, $\partial(W'_\lambda \cup \mathbb{D})$ 'close to $\pi/2$ '-in the sense that we can assume them to be positive and smaller then π - (we may take parabolic-like restrictions). To obtain a diffeomorphic motion of the annulus $A_{h_{\lambda_0}}$ we construct diffeomorphic motions of the quasidisks $Q_{h_{\lambda_0}}$ and $\Delta_{h_{\lambda_0}}$ using the Douady-Earle extension.

Let $\psi_{Q_\lambda} : Q_{h_\lambda} \rightarrow \mathbb{D}$, $\lambda \in M_f$ be a family of Riemann maps depending continuously on λ , and let $\phi_{Q_\lambda} : \mathbb{D} \rightarrow Q_{h_\lambda}$ be the family of inverse maps. Then ϕ_{Q_λ} depends continuously on λ and extends continuously to the boundaries, and since ∂Q_{h_λ} is a quasicircle the family $\phi_{Q_\lambda} : \mathbb{S}^1 \rightarrow \partial Q_{h_\lambda}$ is quasisymmetric in z and continuous in (λ, z) .

Hence the family of quasisymmetric homeomorphisms $\varphi_{Q_\lambda} := \phi_{Q_\lambda}^{-1} \circ \alpha_\lambda \circ \hat{\tau} \circ \alpha_{\lambda_0}^{-1} \circ \phi_{Q_{\lambda_0}} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ continuous in (λ, z) , extends (cfr. [DE]) to a family of quasiconformal maps $\Phi_{Q_\lambda} : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$, which are C^1 diffeomorphism on \mathbb{D} , continuous in (λ, z) . Then $\hat{\Psi}_{Q_\lambda} := \phi_{Q_\lambda} \circ \Phi_{Q_\lambda} \circ \psi_{Q_{\lambda_0}} : Q_{h_{\lambda_0}} \rightarrow Q_{h_\lambda}$ is a family of quasiconformal maps which are C^1 diffeomorphisms, depending continuously on (λ, z) (see Fig.1).

On the other hand, let $\psi_{\Delta_{h_\lambda}} : \Delta_{h_\lambda} \rightarrow \mathbb{D}$ be a family of Riemann maps depending continuously on λ , and let $\phi_{\Delta_{h_\lambda}} : \mathbb{D} \rightarrow \Delta_{h_\lambda}$ be the family of inverse maps. Then $\phi_{\Delta_{h_\lambda}}$ depends continuously on λ , and it extends continuously to the boundary. Moreover, since $\partial \Delta_{h_\lambda}$ is a quasicircle, the restriction $\phi_{\Delta_{h_\lambda}} : \mathbb{S}^1 \rightarrow \partial \Delta_{h_\lambda}$ is quasisymmetric.

Define the family of homeomorphisms $\varphi_{\Delta_{h_\lambda}} := \phi_{\Delta_{h_\lambda}}^{-1} \circ \alpha_\lambda \circ \hat{\tau} \circ \alpha_{\lambda_0}^{-1} \circ \phi_{\Delta_{h_{\lambda_0}}} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ continuous in (λ, z) . How we saw before, for every $\lambda \in M_f$, the map $\alpha_\lambda \circ \hat{\tau} \circ \alpha_{\lambda_0}^{-1}$ is a quasisymmetric homeomorphism, hence for every $\lambda \in M_f$ the map $\varphi_{\Delta_{h_\lambda}} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a quasisymmetric homeomorphism. Therefore the family $\varphi_{\Delta_{h_\lambda}}$ extends by the Douady-Earle extension (cfr. [DE]) to a family of quasiconformal maps $\Phi_{\Delta_{h_\lambda}} : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$, real-analytic diffeomorphisms on \mathbb{D} , continuous in (λ, z) .

Therefore, the family $\hat{\Psi}_{\Delta_\lambda} := \phi_{\Delta_\lambda} \circ \Phi_{\Delta_{h_\lambda}} \circ \psi_{\Delta_{\lambda_0}} : \Delta_{h_{\lambda_0}} \rightarrow \Delta_{h_\lambda}$ is a continuous (in both (λ, z)) family of quasiconformal maps which are C^1 diffeomorphisms, and which extends to $\alpha_\lambda \circ \Phi_\lambda \circ \alpha_{\lambda_0}^{-1}$ on $\gamma_{h_\lambda+}$ and $\gamma_{h_\lambda-}$.

Hence we can define a diffeomorphic motion $\hat{\tau}' : A_{\lambda_0} \times M_f \rightarrow A_\lambda$ as follows:

$$\hat{\tau}'(z) = \begin{cases} \alpha_\lambda^{-1} \circ \hat{\Psi}_{Q_\lambda} \circ \alpha_{\lambda_0} & \text{on } Q_{\lambda_0} \\ \alpha_\lambda^{-1} \circ \hat{\Psi}_{\Delta_{h_\lambda}} \circ \alpha_{\lambda_0} & \text{on } \Delta_{\lambda_0} \\ \Phi_\lambda & \text{on } \partial Q_{\lambda_0} \cap \partial \Delta_{\lambda_0} = \gamma_{\lambda_0+}[1/d, 1] \cup \gamma_{\lambda_0-}[-1/d, -1] \end{cases}$$

where $\Phi : \Lambda \times \gamma_{\lambda_0} \rightarrow \mathbb{C}$ is the holomorphic motion of the dividing arcs (see 1.1). The family $\hat{\tau}' : A_{\lambda_0} \times M_f \rightarrow A_\lambda$ is a family of quasiconformal maps which are C^1 diffeomorphisms, and which are continuous as a function of (λ, z) .

We can now define a tubing which is quasiconformal and a C^1 -diffeomorphism in z , and continuous in (λ, z) .

Definition 3.1. Let us call **diffeomorphic tubing** the map $\widehat{T} := \widehat{\tau}' \circ \widetilde{\Psi} : M_f \times A \rightarrow A_\lambda$, where $\widetilde{\Psi} : A \rightarrow A_{\lambda_0}$ is the quasiconformal C^1 diffeomorphism constructed in 2.0.2.

3.2.2 Continuity of χ on M_f .

Proposition 3.2. *On the open set \mathring{M}_f both ϕ_λ and P_{A_λ} depend continuously on λ .*

Proof. The proof follows the one in [DH]. We write it here for completeness. Let $U \subset \mathbb{C}$ be compact, (μ_n) be a sequence of Beltrami forms on U and μ be another Beltrami form on U , then if:

1. $\exists m < 1 : \|\mu\|_\infty \leq m$ and $\|\mu_n\|_\infty \leq m \forall n$,
2. $\mu_n \xrightarrow{L^1} \mu$,

the family of integrating maps ϕ_λ converges to ϕ uniformly on \mathbb{C} (see [Hu], pg.154).

Since $\|\mu_n\|_\infty \leq m \forall n$ on any compact subset of Λ (see 2.0.4), the continuity of the straightening map (and thus of P_{A_λ}) follows by proving that

$$\mu_\lambda \xrightarrow{L^1} \mu_{\lambda_0} \text{ as } \lambda \rightarrow \lambda_0.$$

Define

$$\hat{\mu}_{\lambda,n}(z) = \begin{cases} \mu_{\lambda,i}(z) & \text{on } A_{\lambda,i} \text{ for } i \leq n \\ 0 & \text{on } U_{\lambda,n} = \widetilde{U}_\lambda \setminus \bigcup_i^{n-1} A_{\lambda,i} \end{cases}$$

Then $\mu_\lambda = \lim_{n \rightarrow \infty} \hat{\mu}_{\lambda,n}$ pointwise. Since $|\mu_\lambda - \mu_{\lambda_0}|_{L^1} \leq |\mu_\lambda - \hat{\mu}_{\lambda,n}|_{L^1} + |\hat{\mu}_{\lambda,n} - \hat{\mu}_{\lambda_0,n}|_{L^1} + |\hat{\mu}_{\lambda_0,n} - \mu_{\lambda_0}|_{L^1}$, in order to prove $\mu_\lambda \xrightarrow{L^1} \mu_{\lambda_0}$ we need to prove that:

$$(a) \quad \hat{\mu}_{\lambda,n} \xrightarrow{L^1} \mu_\lambda \text{ as } n \rightarrow \infty$$

$$(b) \quad \hat{\mu}_{\lambda,n} \xrightarrow{L^1} \hat{\mu}_{\lambda_0,n} \text{ as } \lambda \rightarrow \lambda_0$$

$$(c) \quad \hat{\mu}_{\lambda_0,n} \xrightarrow{L^1} \mu_{\lambda_0} \text{ as } n \rightarrow \infty$$

Clearly (a) \Rightarrow (c), hence we have to prove (a) and (b). Let us start by proving (b).

(b) On Δ_λ the beltrami forms $\hat{\mu}_{\lambda,n}$ and $\hat{\mu}_{\lambda,0}$ coincide (by definition of \widetilde{f}_λ), and on Ω_λ the pull back is done by f_λ , which depends holomorphically on λ . Hence to show that for each n , $\hat{\mu}_{\lambda,n}$ depends continuously on λ in the L^1 norm, it is enough to show that $\hat{\mu}_{\lambda,0}$ depends continuously on λ in the L^1 norm, i.e.

$$\int |\hat{\mu}_{\lambda,0} - \hat{\mu}_{\lambda_0,0}| \xrightarrow{\lambda \rightarrow \lambda_0} 0.$$

Since:

$$\widehat{T}_\lambda : (A, \mu_0) \rightarrow (A_\lambda, \hat{\mu}_{\lambda,0})$$

we can compute

$$\hat{\mu}_{\lambda,0}(z) = (\hat{T}_\lambda^{-1})^*(\mu_0)(z) = \frac{\partial \bar{z} \hat{T}_\lambda^{-1}(z)}{\partial z \hat{T}_\lambda^{-1}(z)}.$$

Since the diffeomorphic tubing \hat{T}_λ is a family of quasiconformal maps which are C^1 -diffeomorphism in z and continuous in (λ, z) , the family of derivatives \hat{T}'_λ and their inverse $(\hat{T}_\lambda^{-1})'$ is continuous in (λ, z) . Therefore $\partial \bar{z} \hat{T}_\lambda^{-1}$ and $\partial z \hat{T}_\lambda^{-1}$ are continuous in (λ, z) , and thus $\hat{\mu}_{\lambda,0}$ depends continuously in (λ, z) . Finally, since $\hat{\mu}_{\lambda,0}$ is continuous and bounded, it depends continuously in λ in the L^1 norm. Therefore $\hat{\mu}_\lambda$ depends continuously in λ in the L^1 norm.

(a) The fact that $\hat{\mu}_{\lambda,n} \xrightarrow{L_1} \mu_\lambda$ as $n \rightarrow \infty$ follows from the fact that the area of $U_{\lambda,n} \setminus K_\lambda$ tends to zero *uniformly* on every compact subset of \mathbb{R} .

Indeed $\hat{\mu}_{\lambda,n}$ and μ_λ are different just on $U_{\lambda,n} \setminus K_\lambda$, hence

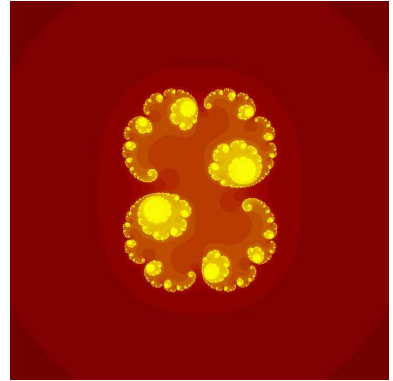
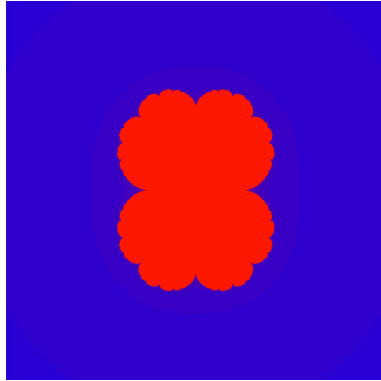
$$|\mu_\lambda - \hat{\mu}_{\lambda,n}|_{L_1} < \sup_{z \in (U_{\lambda,n} \setminus K_\lambda)} |\mu_\lambda(z) - \hat{\mu}_{\lambda,n}(z)| * \text{area}(U_{\lambda,n} \setminus K_\lambda).$$

Since $\hat{\mu}_{\lambda,n} = 0$ on $U_{\lambda,n}$, and $\sup_z |\mu_\lambda| = \|\mu_\lambda\|_\infty = \|\mu_{\lambda,0}\|_\infty < 1$, we obtain the following bound:

$$|\mu_\lambda - \hat{\mu}_{\lambda,n}|_{L_1} < 1 * \text{area}(U_{\lambda,n} \setminus K_\lambda).$$

Therefore, $\text{Area}(U_{\lambda,n} \setminus K_\lambda) \xrightarrow{n \rightarrow \infty} 0$ locally uniformly on \mathbb{R} implies $\hat{\mu}_{\lambda,n} \xrightarrow{L_1} \mu_\lambda$.

Remark 3.1. The area of $U_{\lambda,n} \setminus K_\lambda$ does not tend to zero on any subset of Λ which intersects F . Indeed for every value of λ for which f_λ has a non persistent parabolic fixed point, the area of $U_{\lambda,n}$ still depends continuously on λ , but the area of K_λ is discontinuous. See the pictures below: the pictures on the left shows the filled Julia set of the map $P_{1/4}(z) = z^2 - 1/4$, which has a non persistent parabolic fixed point, and the picture on the right shows the filled Julia set of the map $P_c(z) = z^2 + c$ with $c = 0.285 + 0.01i$.



Choose $\lambda_0 \in \mathring{M}_f$, let $W(\lambda_0)$ be a neighborhood of λ_0 in \mathring{M}_f and consider a dynamic holomorphic motion

$$\tau(\lambda, z) : W(\lambda_0) \times U(J_{\lambda_0}) \rightarrow \mathbb{C}$$

$$(\lambda, z) \rightarrow z_\lambda$$

extension to a neighborhood $U(J_{\lambda_0})$ of J_{λ_0} of the dynamic holomorphic motion of the Julia set constructed locally on R in 1.2 (see 1.3). Define $U(J_\lambda) = \tau_\lambda(U(J_{\lambda_0}))$, $B_\lambda = U(J_\lambda) \cup K_\lambda$ and $B'_\lambda = f_\lambda^{-1}(B_\lambda)$. Then, for every $\lambda \in W(\lambda_0)$, $(f_\lambda, B'_\lambda, B_\lambda, \gamma_\lambda)$ is a parabolic-like restriction of $(f_\lambda, U'_\lambda, U_\lambda, \gamma_\lambda)$. Set $V_{\lambda,0} = B_\lambda \setminus \tilde{\Omega}'_{B_\lambda}$, $V_{\lambda,n} = \tilde{f}_\lambda^{-n}(V_{\lambda,0}) \cap B_\lambda$, $B_{\lambda,n} = B_\lambda \setminus \bigcup_{i=0}^{n-1} V_{\lambda,i}$.

There exists a neighborhood $W(\lambda_0)'$ of λ_0 with compact closure in $W(\lambda_0)$ and $p \in \mathbb{N}$ such that $U_{\lambda,p} \subset B_\lambda$ for all $\lambda \in W(\lambda_0)'$. We then obtain $U_{\lambda,p+n} \subset B_{\lambda,n}$.

Let us define $m_n(\lambda) = \text{area}(B_{\lambda,n} \setminus K_\lambda)$. Clearly $\text{area}(B_{\lambda,n} \setminus K_\lambda) \xrightarrow{u} 0$ implies $\text{area}(U_{\lambda,n} \setminus K_\lambda) \xrightarrow{u} 0$.

Since $U(J_\lambda) = \tau_\lambda(U(J_{\lambda_0}))$, $B_\lambda = U(J_\lambda) \cup K_\lambda$ and the holomorphic motion $\tau(\lambda, z) : W(\lambda_0) \times U(J_{\lambda_0}) \rightarrow \mathbb{C}$ is dynamic (hence $\tau_\lambda(V_{\lambda_0,n}) = V_{\lambda,n}$), we can write

$$m_n(\lambda) = \int_{B_{\lambda_0,n} \setminus K_{\lambda_0}} |Jac(\tau_\lambda)| dx dy.$$

Clearly $m_n \rightarrow 0$ pointwise. Set $[D\tau(z)] : T_z U_{\lambda_0} \rightarrow T_{\tau_\lambda(z)} U_\lambda$, $[D\tau(z)](u) = \frac{\partial \tau_\lambda(z)}{\partial z}(u) + \frac{\partial \tau_\lambda(z)}{\partial \bar{z}}(\bar{u})$, then $\|D\tau_\lambda\| = \sup_{\|z\|=1} \|D\tau_\lambda(z)\| = |\frac{\partial \tau_\lambda}{\partial z}| + |\frac{\partial \tau_\lambda}{\partial \bar{z}}|$ and

$$Jac\tau_\lambda \leq \|D\tau_\lambda\|^2 \leq K Jac\tau_\lambda$$

where $K = \frac{1+|\mu_\lambda|}{1-|\mu_\lambda|} > 1$ and $\mu_\lambda = (\tau_\lambda)^* \mu_0$. Since τ_λ is holomorphic in λ , the sequence

$$n_n(\lambda) = \int_{B_{\lambda_0,n} \setminus K_{\lambda_0}} \|D\tau_\lambda\|^2 dx dy,$$

is subharmonic. Since

$$m_n \leq n_n \leq K m_n,$$

we have that

$$\frac{1}{K} n_n \leq m_n \leq n_n.$$

Since $m_n \rightarrow 0$ pointwise, then $n_n \rightarrow 0$ pointwise. The sequence $n_n \rightarrow 0$ decreases, hence it is uniformly bounded on any compact set; and thus it converges in L^1_{loc} [Hö]. Since the limit function is constant, the sequence $n_n \rightarrow 0$ converges to zero uniformly on any compact subset of $W(\lambda_0)'$, and thus $m_n(\lambda) \rightarrow 0$ uniformly on any compact subset of $W(\lambda_0)'$.

Therefore on \dot{M}_f the straightening map ϕ_λ converges uniformly to ϕ_{λ_0} as $\lambda \rightarrow \lambda_0$, which implies that $P_{A_\lambda} := \phi_\lambda \circ \tilde{f}_\lambda \circ \phi_\lambda^{-1}$ is continuous in λ on \dot{M}_f . \square

3.2.3 Continuity of χ on Λ .

Proposition 3.3. *The proof is very inspired by the one in [DH].*

Suppose $A_1, A_2 \in \mathbb{C}$, with $B_1 = 1 - (A_1)^2 \in \partial M_1$. If the map P_{A_1} and P_{A_2} are quasiconformally conjugate, then $(A_1)^2 = (A_2)^2$.

Let $\phi : U \rightarrow V$ be a quasiconformal conjugacy between P_{A_1} and P_{A_2} . If K_{P_1} is of measure zero, then ϕ is a hybrid conjugacy and the result follows from Theorem [LL].

Therefore let us consider the case K_{P_1} not of measure zero. Define on U the following Beltrami form:

$$\tilde{\mu}(z) := \begin{cases} (\phi)^* \mu_0 & \text{on } K_{P_1} \\ 0 & \text{on } \mathbb{C} \setminus K_{P_1} \end{cases}$$

Since ϕ is quasiconformal, $\|\tilde{\mu}\|_\infty = k < 1$. Therefore we can define on U a new Beltrami form $\mu_t = \tilde{\mu}t$, $\forall t : |t| < 1/k$, and we have $\|\mu_t\|_\infty < 1$. Note that the Beltrami form μ_t depends holomorphically on t . Let

$$\Phi_t : \mathbb{C} \rightarrow \mathbb{C}$$

be the family of quasiconformal maps such that $(\Phi_t)^* \mu_0 = \mu_t$, $\Phi_t(\infty) = \infty$, $\Phi_t(-1) = -1$ and $\Phi_t(0) = 0$. Then the family Φ_t depends holomorphically on t (since μ_t depends holomorphically on t), $\Phi_1 = \phi$ and $\Phi_0 = Id$.

The family of holomorphic maps $F_t = \Phi_t \circ P_{A_1} \circ \Phi_t^{-1}$ has the form $F_t(z) = z + 1/z + A(t)$ (since it is a family of quadratic rational maps with a parabolic fixed point at $z = \infty$) and it depends holomorphically on t (since the family Φ_t depends holomorphically on t).

Then the map $A : t \rightarrow A(t)$ is holomorphic with $A(0) = A_1$, hence $\alpha : t \rightarrow B(t) = 1 - A^2(t)$ is a holomorphic map, with $\alpha(0) = B_1 \in \partial M_1$. Since $\alpha(t)$ is holomorphic, it is either an open map or it is constant. If $\alpha(t)$ is open, since $\alpha(0) \in \partial M_1 \subset M_1$, there exists a neighborhood W of 0 such that $\alpha(W) \subset M_1$. Since $\alpha(0) \in \partial M_1$, it is impossible. Hence the map $\alpha(t)$ is constant, and $\alpha(t) = B_1$, $\forall t$. In particular, for $t = 1$, we have $\alpha(1) = B_1$, and $F_1 = P_{A_1}$.

Therefore the map $\phi \circ \Phi_1^{-1}$ is a quasiconformal conjugacy between P_{A_1} and P_{A_2} , with $(\phi \circ \Phi_1^{-1})^* \mu_0 = \mu_0$ on K_{P_1} . Hence the map $\phi \circ \Phi_1^{-1}$ is a hybrid conjugacy between P_{A_1} and P_{A_2} , and then $(A_1)^2 = (A_2)^2$ by [LL].

Proposition 3.4. *Choose $\lambda_0 \in \Lambda$ and let (λ_n) be a sequence in Λ converging to λ_0 . Then there exists a subsequence $(\lambda_k^*) = (\lambda_{k_n})$ such that the maps $P_{A_k^*}$ converge to a map \widetilde{P}_A and such that the $\phi_{\lambda_k^*}$ converge uniformly on every compact subset of U_{λ_0} to a quasi-conformal equivalence $\tilde{\phi}$ between f_{λ_0} and \widetilde{P}_A .*

Proof. The proof follows the one in [DH]. We write it here for completeness. Choose $\lambda_0 \in \Lambda$ and let (λ_n) be a sequence in Λ converging to λ_0 . Let ϕ_{λ_n} be a family of hybrid conjugacies between f_{λ_n} and P_{A_n} . The ϕ_{λ_n} are quasi-conformal with the same dilatation ratio, as they are uniformly quasiconformal, and hence they form an equicontinuous family (see [A] pg.49, or [Hu] pg. 129).

Since the ϕ_{λ_n} are equicontinuous, there exists a subsequence $\phi_{\lambda_k^*}$ which converges to some quasiconformal limit map $\tilde{\phi}$ when $\lambda \rightarrow \lambda_0$. Hence:

$$\begin{aligned} f_{\lambda_n} &\xrightarrow{\lambda \rightarrow \lambda_0} f_{\lambda_0}, \\ \phi_{\lambda_k^*} &\xrightarrow{\lambda \rightarrow \lambda_0} \tilde{\phi}. \end{aligned}$$

Therefore

$$P_{A_k^*} \xrightarrow{\lambda \rightarrow \lambda_0} \widetilde{P}_A,$$

where $\widetilde{P}_A = \tilde{\phi} \circ f_{\lambda_0} \circ \tilde{\phi}^{-1}$.

Remark 3.2. Note that the limit $\tilde{\phi}$ of a subsequence $\phi_{\lambda_k^*}$ of hybrid conjugacies between the maps $f_{\lambda_k^*}$ and $P_{A_k^*}$ is just a quasiconformal conjugacy between the limit maps f_{λ_0} and \tilde{P} . Indeed, when $\lambda_n \rightarrow \lambda_0$ with $\lambda_n \notin M_f$ and $\lambda_0 \in \partial M_f$, the filled Julia sets of the maps belonging to the subsequences $f_{\lambda_k^*}$ and $P_{A_k^*}$ are without interior, while the filled Julia set of limit maps f_{λ_0} and \tilde{P} may have interior: $\overline{\partial}\phi_{\lambda_k^*} = 0$ on a measure zero set does not imply $\overline{\partial}\tilde{\phi} = 0$ on a set with positive measure.

□

Proposition 3.5. The map $\chi : \Lambda \rightarrow \mathbb{C}$ is continuous.

Proof. The proof follows the one in [DH]. We write it here for completeness. By prop.3.2 the map χ is continuous on R . Let $\lambda_n \in \Lambda$ be a sequence converging to a point $\lambda_0 \in F$. Let us show that we can choose a subsequence $\lambda_n^* = \lambda_{n_k}$ such that the map $\chi(\lambda_n^*)$ converges to $\chi(\lambda_0)$.

Let P_{A_0} be a representative of the class in $Per_1(1)$ hybrid equivalent to f_{λ_0} , and let us show first that $\chi(\lambda_0)$ belongs to ∂M_1 . Let λ_m be a sequence in I converging to λ_0 . By lemma 3.4 there exists a subsequence $\lambda_m^* = \lambda_{m_k}$ such that $P_{\chi(\lambda_m^*)}$ converges to a $P_{\chi(\hat{m})}$ quasiconformally equivalent to f_{λ_0} . For all m the map f_{λ_m} has an indifferent periodic point, hence $P_{A_{\chi(\lambda_m)}}$ has an indifferent periodic point and thus $\chi(\lambda_m)$ belongs to ∂M_1 . Therefore, the limit $\chi(\hat{m}) = \lim(\chi(\lambda_m))$ belongs to ∂M_1 . Hence $P_{\chi(\hat{m})}$ is quasiconformally equivalent to P_{A_0} and $\chi(\hat{m}) \in \partial M_1$, thus by Prop.3.3 $P_{\chi(\hat{m})}$ and P_{A_0} are in the same class. Therefore $\chi(\lambda_0)$ belongs to ∂M_1 .

Now let $(\lambda_n) \in \Lambda$ be a sequence converging to λ_0 . By the previous lemma, there exists a subsequence (λ_k^*) such that:

$$\phi_{\lambda_k^*} \xrightarrow{\lambda \rightarrow \lambda_0} \tilde{\phi},$$

and $\tilde{\phi}$ is a quasiconformal conjugacy between f_{λ_0} and $P_{\tilde{A}} = \tilde{\phi} \circ f_{\lambda_0} \circ \tilde{\phi}^{-1}$. Therefore f_{λ_0} is quasiconformally conjugate to both $P_{\tilde{A}}$ and P_{A_0} . Hence by proposition 3.3, $P_{\tilde{A}}$ and P_{A_0} are in the same class of $Per_1(1)$. Therefore the subsequence $\chi(\lambda_n^*)$ converges to the limit $\chi(\lambda_0)$, and finally the map χ is continuous. □

3.3 Analicity of χ on the interior of M_f .

In this section we prove that the map $\chi : \Lambda \rightarrow \mathbb{C}$ depends analytically on λ for $\lambda \in \mathring{M}_f$ (Corollary 3.1), and that for all $\hat{A} \in M_1 \setminus \{1\}$, $\chi^{-1}(\hat{A})$ is a complex analytic subset of M_f .

Proposition 3.6. Let $f = (f_\lambda : U'_\lambda \rightarrow U_\lambda)$ be an analytic family of parabolic-like maps of degree 2 parametrized by Λ and $g = (g_\iota : V'_\iota \rightarrow V_\iota)$ be an analytic family of parabolic-like maps of degree 2 parametrized by I , where $\Lambda \approx \mathbb{D}$, $I \approx \mathbb{D}$. Let W_1 be a connected component of R_f contained in M_f , W_2 a connected component of R_g contained in M_g . Then the set $\Gamma \subset W_1 \times W_2$ of those (λ, ι) for which f_λ and g_ι are hybrid equivalent is a complex-analytic subset of $W_1 \times W_2$.

Proof. The proof follows the one in [DH], with the differences given by the geometry of our setting.

Choose $\iota_0 \in W_2$ and let $T_g : I \times A \rightarrow A_{\iota}$ be a holomorphic tubing of $(g_{\iota})_{\iota \in I}$ (see 2.0.3). This defines a dividing arc $\tilde{\gamma}$ and a fundamental annulus A for h_2 (see 3.2). Choose $\lambda_0 \in W_1$ and let Λ' be a neighborhood of λ_0 in W_1 . For every λ in Λ' , let $\Xi_{\lambda+}$ and $\Xi_{\lambda-}$ be the repelling petals (for the parabolic fixed point z_{λ} of f_{λ}) where $\gamma_{\lambda+}$ and $\gamma_{\lambda-}$ respectively reside, and let $\phi_{\lambda} : \Xi_{\lambda} \rightarrow \mathbb{H}_{\pm}$ be repelling Fatou coordinates. If necessary, take a parabolic-like restriction of f_{λ_0} such that, for all $\lambda \in \Lambda'$, $U_{\lambda_0} \subseteq U_{\lambda}$ and there exists U_+ neighborhood of $\gamma_{\lambda+}(1) \cap \partial U_{\lambda_0}$ in ∂U_{λ_0} and a horizontal strip in each Fatou coordinate plane, such that $\phi_{\lambda+}(U_+)$ crosses this strip once and without horizontal slopes, and there exists U_- neighborhood of $\gamma_{\lambda-}(-1) \cap \partial U_{\lambda_0}$ in ∂U_{λ_0} and a horizontal strip in each Fatou coordinate plane, such that $\phi_{\lambda-}(U_-)$ crosses this strip once and without horizontal slopes. Redefine for all $\lambda \in \Lambda'$ the set $U'_{\lambda} = f_{\lambda}^{-1}(U_{\lambda_0})$. Let ϕ_h be Fatou coordinates for h_2 , and redefine the dividing arcs γ_{λ} as $\gamma_{\lambda} := \phi_{\lambda}^{-1} \circ \phi_h(\tilde{\gamma})$. Hence for all $\lambda \in \Lambda'$, the new dividing arcs are isotopic to the original ones. Note that this redefines the holomorphic motion of the dividing arcs Φ_{λ} on Λ' as $\Phi_{\lambda}(\gamma_{\lambda_0}) = \phi_{\lambda}^{-1} \circ \phi_{\lambda_0}(\gamma_{\lambda_0})$. Then for all $\lambda \in \Lambda'$, $f_{\lambda} : U'_{\lambda} \rightarrow U_{\lambda_0}$ is a parabolic-like restriction of $(f_{\lambda}, U'_{\lambda}, U_{\lambda}, \gamma_{\lambda})$. Note that the boundaries still move holomorphically. Indeed, the range is constant and the domains move holomorphically since f_{λ} depends holomorphically on λ . Let $\Psi_{\lambda_0} : A \rightarrow A_{\lambda_0}$ be a quasiconformal C^1 diffeomorphism whose restriction $\Psi_{\lambda_0} : \tilde{\gamma} \rightarrow \gamma_{\lambda_0}$ conjugates dynamics (the construction of Ψ_{λ_0} is given by 2.0.2, the only difference is that $\Phi_{\Delta_{\lambda_0}}$ is a quasiconformal map C^1 diffeomorphism which here extends on γ_{λ} by $\phi_h^{-1} \circ \phi_{\lambda}(\gamma_{\lambda})$). Let $\hat{\tau} : \Lambda' \times \partial(U_{\lambda_0} \setminus \Omega'_{\lambda_0}) \rightarrow \partial(U_{\lambda_0} \setminus \Omega'_{\lambda})$ be the holomorphic motion given by $\hat{\tau}|_{\partial U_{\lambda_0}} = Id$, $\hat{\tau}|_{\gamma_{\lambda_0}} = \Phi_{\lambda}$, and $\hat{\tau}|_{\partial U'_{\lambda_0} \cap \partial \Omega_{\lambda}} = f_{\lambda}^{-1} \circ f_{\lambda_0}$ (where f_{λ}^{-1} is the branch which preserves the dynamics on the dividing arcs). Finally let $\bar{\tau} : \Lambda' \times A_{\lambda_0} \rightarrow A_{\lambda}$ be the restriction to the fundamental annulus A_{λ_0} of the extension (given by the Slodkowski theorem) to $\hat{\mathbb{C}}$ of the holomorphic motion $\hat{\tau}$. Hence $T_f := \bar{\tau} \circ \Psi_{\lambda_0} : \Lambda' \times A \rightarrow A_{\lambda}$ is a holomorphic tubing.

Define for any $(\lambda \times \iota) \in \Lambda' \times I$ the map (see Figure 2):

$$\delta_{(\lambda, \iota)} := T_g \circ T_f^{-1} : I \times \Lambda' \times A_{\lambda} \rightarrow A_{\iota},$$

and define for any $\iota \in I$ the map:

$$\tilde{\delta}_{(\iota)} := \delta_{\lambda, \iota} \circ \bar{\tau}_{\lambda} = T_g \circ \Psi_{\lambda_0}^{-1} : I \times A_{\lambda_0} \rightarrow A_{\iota}$$

In order to prove that the set Γ of those (λ, ι) for which f_{λ} and g_{ι} are hybrid equivalent is a complex-analytic subset of $W_1 \times W_2$ we will now prove that:

1. For every $(\lambda, \iota) \in \Lambda' \times I$, the map $\delta_{(\lambda, \iota)}$ defines an almost complex structure on U_{λ_0} which depends *holomorphically* on (λ, ι) ;
2. the set of (λ, ι) for which the map $\delta_{(\lambda, \iota)} : A_{\lambda} \rightarrow A_{\iota}$ extends to a holomorphic map $\alpha : U_{\lambda_0} \rightarrow U_{\iota}$ which conjugates f_{λ} and g_{ι} equals Γ ;
3. the set of (λ, ι) for which the map $\delta_{(\lambda, \iota)} : A_{\lambda} \rightarrow A_{\iota}$ extends to a holomorphic map $\alpha : U_{\lambda_0} \rightarrow U_{\iota}$ is a complex analytic subset of $\Lambda' \times W_2$.

Remark 3.3. *By construction, for every $\lambda \in \Lambda'$ the range of the parabolic-like restriction of f_{λ} is U_{λ_0} . The fundamental annulus of f_{λ} is still dependent on λ , since it is $A_{\lambda} = U_{\lambda_0} \setminus \overline{\Omega'_{\lambda}}$.*

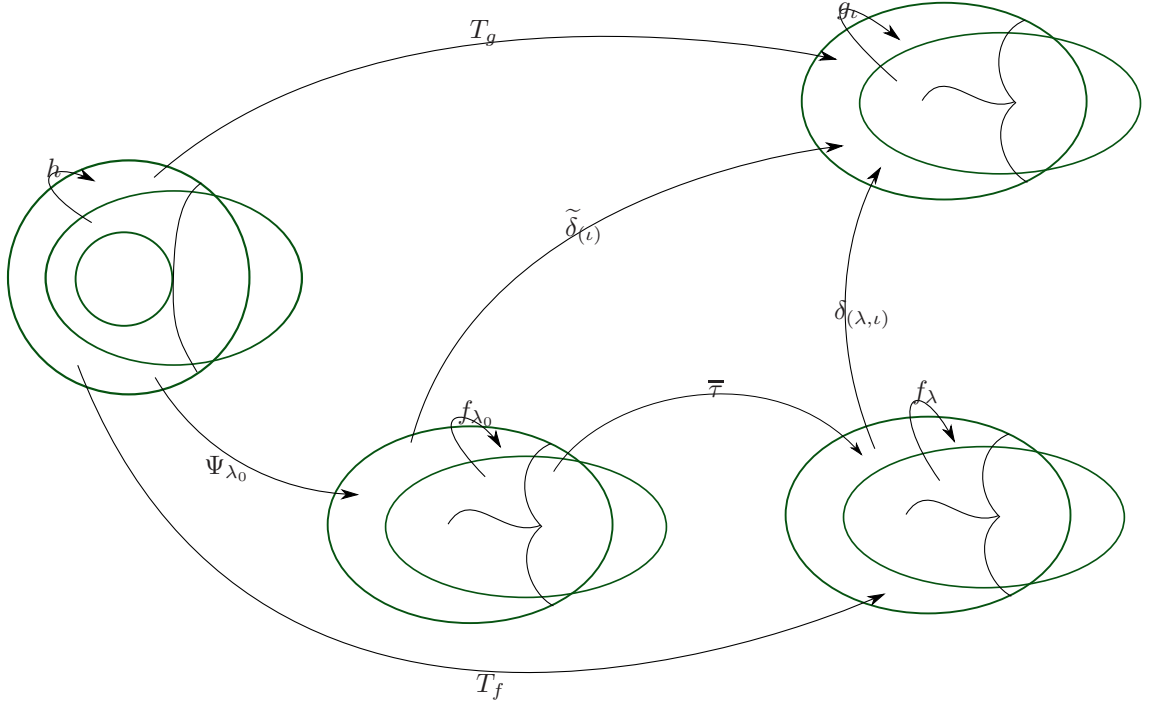


Figure 2: Construction of the maps $\delta_{(\lambda, \iota)} := T_{g_\iota} \circ T_{f_\lambda}^{-1} : \Lambda' \times I' \times A_{f_\lambda} \rightarrow A_{g_\iota}$ and $\tilde{\delta}_{(\iota)} := \gamma_{(\lambda, \iota)} \circ \tau_\lambda = T_{g_\iota} \circ T_{f_{\lambda_0}}^{-1} : \Lambda' \times A_{f_{\lambda_0}} \rightarrow A_{g_\iota}$.

(1) For every $\lambda \in \Lambda' \setminus \lambda_0$ define on U_{λ_0} the following Beltrami form:

$$\nu_{(\lambda, \iota)}(z) := \begin{cases} \nu_{\lambda, \iota, 0} = (\delta_{(\lambda, \iota)})^* \mu_0 & \text{on } A_\lambda \\ (f_{(\lambda, \iota)}^n)^* \nu_{\lambda, \iota, 0} & \text{on } A_{f_\lambda, n} = \tilde{f}_{(\lambda, \iota)}^{-n}(A_{f_\lambda}) \\ 0 & \text{on } K_{f_\lambda} \end{cases}$$

where in this case the map $\tilde{f}_{(\lambda, \iota)}$ which spreads the Beltrami form $\nu_{\lambda, \iota, 0}$ depends on both (λ, ι) , and it is defined as follows:

$$\tilde{f}_{(\lambda, \iota)}(z) = \begin{cases} \delta_{(\lambda, \iota)}^{-1} \circ \tilde{g}_\iota \circ \delta_{(\lambda, \iota)} & \text{on } \tilde{\Delta}_\lambda' \\ f_\lambda & \text{on } \tilde{\Omega}_\lambda' \end{cases}$$

where \tilde{g}_ι is as in 2.0.2. Note that, even if $\tilde{f}_{(\lambda, \iota)}$ depends on both (λ, ι) , its range is still \tilde{U}_{λ_0} . Indeed, $T_{f_\lambda}^{-1}(\tilde{\Delta}_\lambda') = \tilde{\Delta}_W'$, $T_{g_\iota}(\tilde{\Delta}_W') = \tilde{\Delta}_\iota'$, $g_\iota(\tilde{\Delta}_\iota') = \tilde{\Delta}_\iota$, $T_{g_\iota}(\tilde{\Delta}_\iota) = \tilde{\Delta}_W$ and finally $T_{f_\lambda}(\tilde{\Delta}_W) = \tilde{\Delta}_\lambda$ (see Fig. 2). For λ_0 define on U_{λ_0} the following Beltrami form:

$$\tilde{\nu}_\iota(z) := \begin{cases} \tilde{\nu}_{\iota, 0} = \tilde{\delta}_{(\iota)}^*(\mu_0) & \text{on } A_{\lambda_0} \\ (\tilde{f}_{(\lambda_0, \iota)}^n)^* \nu_{\iota, 0} & \text{on } A_{f_{\lambda_0}, n} = \tilde{f}_{\lambda_0}^{-n}(A_{f_{\lambda_0}}) \\ 0 & \text{on } K_{f_{\lambda_0}} \end{cases}$$

Let us show that for every $z \in U_{\lambda_0}$ the map

$$\tilde{\nu}_\iota(z) : I \longrightarrow L^\infty(U_{\lambda_0})$$

$$\iota \rightarrow (z \rightarrow \tilde{\nu}_{(\iota)}(z))$$

is complex analytic in ι . Indeed, $\tilde{\delta}_\iota^*(\mu_0) = (T_{g_\iota} \circ \Psi_{\lambda_0}^{-1})^*(\mu_0) = (\Psi_{\lambda_0})_*(T_{g_\iota}^* \mu_0)$, and calling $\mu_\iota = T_{g_\iota}^* \mu_0$, the map

$$\iota \rightarrow \mu_\iota(z)$$

is complex analytic in ι for every $z \in A$, since T_g is a holomorphic tubing. Since the map Ψ_{λ_0} does not depend on λ , the family of Beltrami forms $\tilde{\nu}_{\iota,0} = \Psi_*(\mu_\iota)$ still depends holomorphically on ι for every $z \in A_{\lambda_0}$. Since on $\tilde{\Omega}'_{\lambda_0}$ the Beltrami form $\tilde{\nu}_{\iota,0}$ is spread by the dynamics of f_{λ_0} (which does not depend on λ nor on ι), and on Δ_{λ_0} it is constant, the Beltrami form $\tilde{\nu}_\iota$ still depends holomorphically on ι for every $z \in \tilde{U}_{\lambda_0}$.

By the Measurable Riemann Mapping theorem with parameters, there exist charts $\tilde{\theta}_\iota : I \times U_{\lambda_0} \rightarrow \mathbb{C}$ depending analytically on ι which integrate the Beltrami form $\tilde{\nu}_\iota$. On the other hand, there exist charts $\theta_{\lambda,\iota}$ which integrate the Beltrami form $\nu_{\lambda,\iota}$, and by construction the following diagram commutes:

$$\begin{array}{ccc} \Lambda' \times I \times A_{\lambda_0} & \xrightarrow{(Id, Id, \bar{\tau}_\lambda)} & \Lambda' \times I \times A_{\lambda_0} \\ \downarrow (Id, \tilde{\theta}_\iota) & & \downarrow \theta_{\lambda,\iota} \\ \Lambda' \times I \times \mathbb{C} & \xrightarrow{(Id, Id, Id)} & \Lambda' \times I \times \mathbb{C} \end{array} \quad (1)$$

The fact that the previous diagram commutes implies that the following diagram commutes:

$$\begin{array}{ccc} \Lambda' \times I \times A_{\lambda_0} & \xrightarrow{(Id, \tilde{\delta}_\iota)} & \Lambda' \times I \times A_\iota \\ \downarrow (Id, \tilde{\theta}_\iota) & & \uparrow \delta_{\lambda,\iota} \\ \Lambda' \times I \times \mathbb{C} & \xleftarrow{\theta_\lambda} & \Lambda' \times I \times A_{\lambda_0} \end{array} \quad (2)$$

hence $\tilde{\delta}_\iota \circ \tilde{\theta}_\iota^{-1} = \delta_{\lambda,\iota} \circ \theta_{\lambda,\iota}^{-1}$. This finally means that, since $\tilde{\delta}_\iota \circ \tilde{\theta}_\iota^{-1}$ depends holomorphically on the parameter ι , the map $\delta_{\lambda,\iota} \circ \theta_{\lambda,\iota}^{-1}$ depends holomorphically on the parameters (λ, ι) , even if $\delta_{\lambda,\iota} \circ \theta_{\lambda,\iota}^{-1}$ depends a priori on (λ, ι) while $\tilde{\delta}_\iota \circ \tilde{\theta}_\iota^{-1}$ depends only on ι .

Let us now return to integrating the Beltrami form $\nu_{\lambda,\iota}$. The next lemma says that, if $(\lambda, \iota) \in \Gamma$ there exists an integrating map $\alpha_{(\lambda,\iota)}$ which conjugates f_λ to g_ι . That is, if $(\lambda, \iota) \in \Gamma$, there exists a map $\alpha_{(\lambda,\iota)} : U_\lambda \rightarrow \mathbb{C}$ such that $\alpha_{(\lambda,\iota)}^*(\mu_0) = \nu_{\lambda,\iota}$, extending $\delta_{(\lambda,\iota)}$ and conjugating f_λ and g_ι .

Lemma 3.2 states that the set of (λ, ι) such that the map $\delta_{(\lambda,\iota)}$ extends to a map $\alpha_{(\lambda,\iota)} : U_\lambda \rightarrow V_\iota$ holomorphic with respect to θ (i.e. the set Γ) is a complex analytic submanifold.

Lemma 3.1. *Recall that $\Gamma := \{(\lambda, \iota) \mid f_\lambda \text{ and } g_\iota \text{ are hybrid equivalent}\}$. For any $(\lambda, \iota) \in \Lambda' \times W_2$ the following conditions are equivalent:*

1. $(\lambda, \iota) \in \Gamma$,

2. there exists an isomorphism

$$\alpha = \alpha_{(\lambda, \iota)} : U_\lambda \rightarrow \mathbb{C}$$

$$(U_\lambda, \nu_{(\lambda, \iota)}) \rightarrow (V_\iota, \mu_0)$$

extending $\delta_{(\lambda, \iota)}$ and conjugating f_λ and g_ι ,

3. there exists a map $\alpha : U_\lambda \rightarrow \mathbb{C}$ holomorphic with respect to $\nu_{(\lambda, \iota)}$ and extending $\delta_{(\lambda, \iota)}$.

Proof. To see that 2 implies 1 it is enough to remark that α is a conjugacy between f_λ and g_ι conformal with respect to $\nu_{(\lambda, \iota)}$, and thus f_λ and g_ι are hybrid equivalent. To see that 2 implies 3 note that an isomorphism with respect to $\nu_{(\lambda, \iota)}$ is a holomorphic map with respect to $\nu_{(\lambda, \iota)}$, and for all $\iota \in I$, $V_\iota \in \mathbb{C}$.

Let us show that 1 implies 2. Let β be a hybrid equivalence between f_λ and g_ι . Define the map $\alpha : U_\lambda \rightarrow V_\iota$ as follows:

$$\alpha(z) := \begin{cases} T_{g_\iota} \circ T_{f_\lambda}^{-1} & \text{on } A_{f_\lambda} \\ \tilde{g}_\iota^{-n} \circ T_{g_\iota} \circ T_{f_\lambda}^{-1} \circ \tilde{f}_\lambda^n & \text{on } B_{f_\lambda, n} \\ \beta & \text{on } K_{f_\lambda} \end{cases}$$

where \tilde{f}_λ , \tilde{g}_ι are as in 2.0.2, and the sets $B_{f_\lambda, n}$ and the lift of the Tubing are constructed in 4.5. Then α is a hybrid conjugacy between f_λ and g_ι which is holomorphic with respect to $\nu_{(\lambda, \iota)}$ outside of K_{f_λ} . By [LL], since $\Lambda' \in M_f$ and $W_2 \in M_g$, the map α is an isomorphism with respect to $\nu_{(\lambda, \iota)}$ conjugating f_λ and g_ι , and it extends $\delta_{(\lambda, \iota)}$ by construction.

To show that 3 implies 2 we need to prove that the map $\alpha : U_\lambda \rightarrow \mathbb{C}$ is an isomorphism, i.e. that it has degree 1. To count the number of preimages under the map α it is enough to calculate the winding number of the image by α of a loop around a point belonging to U_λ . Since α is a holomorphic extension of $\delta_{(\lambda, \iota)}$, this is the winding number of the image by $\delta_{(\lambda, \iota)}$ of a loop around a point in A_λ , which is 1 since $\delta_{(\lambda, \iota)} = T_{g_\iota} \circ T_{f_\lambda}^{-1}$. \square

Lemma 3.2. *The set of (λ, ι) such that the map $\delta : A_\lambda \rightarrow A_\iota$ extends to a map $\alpha_{\lambda, \iota} : U_\lambda \rightarrow V_\iota$ holomorphic with respect to $\theta_{(\lambda, \iota)}$, is a complex analytic subset of $\Lambda' \times W_2$.*

Proof. The set of (λ, ι) such that the map $\delta : A_\lambda \rightarrow A_\iota$ extends to a map $\alpha_{\lambda, \iota} : U_\lambda \rightarrow V_\iota$ holomorphic with respect to $\theta_{(\lambda, \iota)}$, is the set of (λ, ι) such that the map $h_{(\lambda, \iota)} := \delta_{\lambda, \iota} \circ \theta_{\lambda, \iota}^{-1} : \theta_{\lambda, \iota}(A_\lambda) \rightarrow A_\iota$ extends to a holomorphic map $\alpha_{\lambda, \iota} \circ \theta_{\lambda, \iota}^{-1} : \theta_{\lambda, \iota}(U_\lambda) \rightarrow V_\iota$.

Let $D_1 \subset\subset D_2$ be C^1 Jordan domains in $\theta_{\lambda, \iota}(A_\lambda)$ such that $\overline{D_2} \setminus D_1 \subset h_{(\lambda, \iota)}^{-1}(A_\iota)$ for all $(\lambda, \iota) \in \Lambda' \times I'$, where $\Lambda' \times I'$ is a neighborhood of (λ_0, ι_0) in $W_1 \times W_2$. Let γ_2 be the anticlockwise oriented Jordan curve which bounds D_2 and let γ_1 be the anticlockwise oriented Jordan curve which bounds D_1 . Define $F(z) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{h_{(\lambda, \iota)}(w)}{w-z} dw$, and $G(z) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{h_{(\lambda, \iota)}(w)}{w-z} dw$. Hence, by the Cauchy integral formula, on $D_2 \setminus \overline{D_1}$, $h_{(\lambda, \iota)}(z) = F(z) - G(z)$.

It is clear that, if $G \equiv 0$, $h_{(\lambda, \iota)}(z) = F(z)$ on $D_2 \setminus \overline{D}_1$, hence $h_{(\lambda, \iota)} = F$ and therefore $h_{(\lambda, \iota)}$ extends holomorphically (and the extension coincides with F) on $\theta_{\lambda, \iota}(U_\lambda)$. On the other hand, if $h_{(\lambda, \iota)}$ extends holomorphically on $\theta_{\lambda, \iota}(U_\lambda)$, by the Cauchy integral formula, on D_2 , $h_{(\lambda, \iota)} = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{h_{(\lambda, \iota)}(w)}{w-z} dw = F(z)$, hence $h_{(\lambda, \iota)} = F$ and thus $G \equiv 0$. Therefore, to prove that $h_{(\lambda, \iota)}$ extends holomorphically on $\theta_{\lambda, \iota}(U_\lambda)$, we need to prove that $G \equiv 0$.

We have:

$$\begin{aligned} G(z) &= \frac{1}{2\pi i} \oint_{\gamma_1} \frac{h_{(\lambda, \iota)}(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{\gamma_1} h_{(\lambda, \iota)}(w) \cdot \frac{1}{w-z} dw = \\ &\text{since } \left(-\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n\right) = \frac{1}{w-z} \\ &= \frac{1}{2\pi i} \oint_{\gamma_1} h_{(\lambda, \iota)}(w) \cdot \left(-\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n\right) dw = \frac{1}{2\pi i} \oint_{\gamma_1} h_{(\lambda, \iota)}(w) \cdot \left(-\sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}}\right) dw = \\ &= -\frac{1}{2\pi i} \left(\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}\right) \oint_{\gamma_1} h_{(\lambda, \iota)}(w) \cdot w^n dw. \end{aligned}$$

Set

$$b_n = -\frac{1}{2\pi i} \oint_{\gamma_1} h_{(\lambda, \iota)}(w) \cdot w^n dw,$$

we obtain

$$G(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}},$$

hence $G \equiv 0$ if and only if $\forall n \geq 0$, $b_n = 0$. Since all the b_n are holomorphic maps in (λ, ι) , this is a complex analytic set. Since the set of (λ, ι) for which $h_{(\lambda, \iota)}$ extends holomorphically to $\theta(U_{\lambda_0})$ is the set of (λ, ι) such that the map $\delta : A_\lambda \rightarrow A_\iota$ extends to a map $\alpha_{\lambda, \iota} : U_\lambda \rightarrow V_\iota$ holomorphic with respect to $\theta_{(\lambda, \iota)}$, we obtain that this is a complex analytic set. \square

Since this set is Γ , we obtain that Γ is a complex analytic subset of $W_1 \times W_2$. \square

Corollary 3.1. *The map $\chi_\lambda : \lambda \rightarrow B$ depends analytically on λ for $\lambda \in \mathring{M}_f$.*

Let us apply the previous proposition to f_λ , $\lambda \in M_f$ and P_A , $B = 1 - A^2 \in M_1 \setminus \{1\}$. Since the graph of χ_λ is the set of (λ, B) for which f_λ is hybrid equivalent to P_A , this is a complex analytic set, and therefore χ_λ is analytic on \mathring{M}_f .

Corollary 3.2. *If $\hat{A} \in M_1 \setminus \{1\}$, then $\chi^{-1}(\hat{A})$ is an analytic subset of M_f .*

Let $B = 1 - \hat{A}^2 \in M_1 \setminus \{1\}$ and consider the constant family $P_A = z + \frac{1}{z} + \hat{A}$, $A \in \mathbb{C}$. Since $B = 1 - \hat{A}^2 \in M_1 \setminus \{1\}$ the subset R for this family is all \mathbb{C} . Let f_λ be an analytic family of parabolic-like maps parametrized by Λ , $\Lambda \approx \mathbb{D}$. Then the set $\{(\lambda, A) \mid f_\lambda \text{ is hybrid equivalent to } P_A\}$ is an analytic subset of $M_f \times \mathbb{C}$ by the previous proposition. Since $\{(\lambda, A) \mid f_\lambda \text{ is hybrid equivalent to } P_A\} = \chi^{-1}(B) \times \mathbb{C}$, we obtain that $\chi^{-1}(B)$ is an analytic subset of M_f .

4 The map $\chi : \Lambda \rightarrow \mathbb{C}$ is a ramified covering from the connectedness locus M_f to $M_1 \setminus \{1\}$.

The aim of this thesis is to prove that the connectedness locus of an analytic family of parabolic-like maps is a ramified covering of $M_1 \setminus \{1\}$. In order to prove that the map $\chi : \Lambda \rightarrow \mathbb{C}$ restricts to a branched covering from the connectedness locus M_f to $M_1 \setminus \{1\}$, we need to prove that: *the map $\chi : M_f \rightarrow M_1 \setminus \{1\}$ has a degree, and that it is a ramified covering*. Then we need to have a notion of degree and of ramified covering. Set $\mathcal{B} = \chi(\Lambda)$.

4.0.1 Degree.

Let X, Y be oriented topological surfaces and $\phi : X \rightarrow Y$ be a continuous map. If ϕ is *proper*, and X, Y are *connected* then ϕ has a degree. Indeed, since ϕ is continuous the induced map $\phi_* : H^2(Y) \rightarrow H^2(X)$ is a homomorphism, since X, Y are surfaces $H_c^2(X) \approx \mathbb{Z}$, $H_c^2(Y) \approx \mathbb{Z}$ (see [H] pg.134), and since ϕ is proper the induced map $\phi_* : H_c^2(Y) \rightarrow H_c^2(X)$ is of the form:

$$\alpha \rightarrow d\alpha$$

for some integer d depending only on ϕ , which is called the *degree of ϕ* , $\deg \phi$.

On the other hand, if X, Y are oriented topological surfaces (or *open* subsets of \mathbb{C}), ϕ is proper, X, Y are connected and for all $y \in Y$, $\phi^{-1}(y)$ is discrete, then $\phi^{-1}(y)$ is finite (since ϕ is proper) and the following formula holds (see [H] pg. 136):

$$\deg \phi = \sum_{x \in \phi^{-1}(y)} i_x(\phi),$$

where $i_x(\phi)$ is the *local degree of ϕ at x* , which is defined as follows: choose neighborhoods U, V of x, y respectively, homeomorphic to \mathbb{D} and such that $\phi(U) \subset V$ and $\{x\} = U \cap \phi^{-1}(y)$. If γ is a loop in $U \setminus \{x\}$ with winding number 1, then $i_x(\phi)$ is the winding number of $\phi(\gamma)$ around y .

Remark 4.1. Note that, if X and Y are closed sets, ϕ is proper, X, Y are connected and for all $y \in Y$, $\phi^{-1}(y)$ is discrete and finite, the equality $\deg \phi = \sum_{x \in \phi^{-1}(y)} i_x(\phi)$ does not hold in general. As a counterexample, set $X = \overline{D}(a, r) \subset \mathbb{C}$, where $a \neq 0$, $|a| < r$, and $\phi(z) = z^2$. Then ϕ is proper because X is compact, $Y = \phi(X)$ is compact because ϕ is continuous, but $\deg \phi \neq \sum_{x \in \phi^{-1}(y)} i_x(\phi)$. On the other hand, let $x_0 \in \mathring{D}(a, r)$, and set $y_0 = \phi(x_0)$. Since $x_0 \in \mathring{D}(a, r)$, $x_0 \cap \partial D(a, r) = \emptyset$, hence $y_0 \cap \phi(\partial D(a, r)) = \emptyset$. Therefore, there exists a neighborhood $V \approx \mathbb{D}$ of y_0 in Y such that $V \cap \phi(\partial D(a, r)) = \emptyset$. If U is the connected component of $\phi^{-1}(V)$ containing y_0 , since $\phi^{-1}(V) \cap \partial D(a, r) = \emptyset$, $U \cap \partial D(a, r) = \emptyset$. The set U contains y_0 , then $U \subset \overline{D}(a, r)$, therefore ϕ restricts to a proper map $\phi|_U : U \rightarrow V$ such that $\deg \phi|_U = \sum_{x \in \phi^{-1}(y) \cap U} i_x(\phi)$. Note that $\deg \phi|_U$ can be one or two. If $\{x_0\} = U \cap \phi^{-1}(y_0)$, $\deg \phi|_U = 2$ only if x_0 is a critical point.

4.0.2 Ramified covering.

Definition 4.1. Suppose X, Y are topological spaces. A map $p : X \rightarrow Y$ is a *covering map* if the following holds.

Every $y \in Y$ has an open neighborhood V such that its preimage $p^{-1}(V)$ can be represented as

$$p^{-1}(V) = \bigcup_{j \in J} U_j,$$

where the U_j , $j \in J$ are disjoint open subsets of X , and all mappings $p|_{U_j} : U_j \rightarrow V$ are homeomorphisms. In particular p is a local homeomorphism.

Definition 4.2. Suppose X, Y are topological spaces. A map $p : X \rightarrow Y$ is a *branched covering map* if every $y \in Y$ has a punctured neighborhood V such that $p : p^{-1}(V) \rightarrow V$ is a covering map.

Definition 4.3. Suppose X, Y are topological spaces, and $p : X \rightarrow Y$ is a branched covering map. A point $x \in X$ is called a *branch point* if there is no neighborhood U of x such that $p|_U$ is injective.

4.1 The map $\chi : \Lambda \rightarrow \mathbb{C}$ is orientation preserving.

Let us remind that so far we proved that the map $\chi : \Lambda \rightarrow \mathbb{C}$ is:

1. continuous,
2. quasiregular on $\Lambda \setminus M_f$ and analytic on \mathring{M}_f ,
3. for all $A \in M_1 \setminus \{1\}$, $\chi^{-1}(A)$ is an analytic subset of M_f .

Hence, if χ is not constant, for every $y \in \mathcal{B}$, $\chi^{-1}(y)$ is discrete. Let us see that, for every $x \in \Lambda$, $i_x(\chi) > 0$.

Theorem 4.4. *Let $\mathbf{f} = (f_\lambda)_{\lambda \in \Lambda}$ be an analytic family of parabolic-like mappings of degree 2. If the mapping $\chi : \Lambda \rightarrow \mathbb{C}$ defined using some holomorphic tubing T of \mathbf{f} is not constant, then for every $x \in \Lambda$, $i_x(\chi) > 0$.*

Proof. The proof follows the proof of topological holomorphy of χ over M in [DH].

We can distinguish 3 cases:

1. $\lambda \in R$, $\chi(\lambda) = B \in \mathring{M}_1$ or $B \in \mathcal{B} \setminus M_1$. Since the map $\chi : \Lambda \rightarrow \mathbb{C}$ is analytic on \mathring{M}_f , and quasiregular on $\Lambda \setminus M_f$, $i_\lambda(\chi) > 0$.
2. $\lambda \in \mathring{M}_f$, $B \in \partial M_1$. Since χ is holomorphic on \mathring{M}_f , χ is open or it is constant. If χ is open there exists a neighborhood Λ' of λ in \mathring{M}_f , such that $\chi(\Lambda') \subset M_1$. Since $B = \chi(\lambda) \in \partial M_1$, this is impossible.
3. $\lambda \in F$, $B \in \partial M_1$. Let \mathbb{D} be a disc in Λ containing λ and no other point of $\chi^{-1}(B)$. Set $\gamma = \partial \mathbb{D}$. Since $\lambda \in F$, there exists in $\mathring{\mathbb{D}}$ a λ' such that $f_{\lambda'}$ has an attracting periodic point and $B' = \chi(\lambda')$ is in the same connected component of B . Hence $i_\lambda(\chi) = \sum_{x \in \phi^{-1}(B') \cap \mathring{\mathbb{D}}} i_x(\chi) > 0$, because every term in the sum is positive, since χ is holomorphic at λ' , and there exists at least one term in the sum, which is λ' .

□

Proposition 4.5. *Let $\mathbf{f} = (f_\lambda)_{\lambda \in \Lambda}$ be an analytic family of parabolic-like mappings of degree 2, and $\chi : \Lambda \rightarrow \mathcal{B}$ be as in (3.2), then:*

1. $\forall \lambda \in \Lambda$ there exist open connected neighborhoods U of λ and V of $B = \chi(\lambda)$, with compact closure in Λ and \mathcal{B} respectively, such that χ restricts to a proper surjective open map $\chi|_U : U \rightarrow V$ of degree $d = i_\lambda(\chi)$;
2. if $d = 1$ the map χ restricts to a homeomorphism $\chi : U \rightarrow V$. More generally, we can write $\chi|_U$ as $\pi \circ \tilde{f}$, where $\pi : \tilde{V} \rightarrow V$ ($\tilde{V} \approx \mathbb{D}$) is a d -fold branched covering of V ramified above B (i.e. a branched covering with branched point \tilde{B} such that $\pi(\tilde{B}) = B$), and $\tilde{f} : U \rightarrow \tilde{V}$ is a homeomorphism. The mapping \tilde{f} is a lift of χ into π , and thus restricts to a homeomorphism of U onto $\pi^{-1}(V)$.

Proof. The proof follows the one in the context of topological holomorphy of [DH].

Let $\lambda \in \Lambda$, $B = \chi(\lambda)$ and U be an open and connected neighborhood of λ , $U \subset \Lambda$ such that $\lambda = \chi^{-1}(B) \cap U$. By 4.0.1, if $\chi|_U$ is proper, then $\deg \chi|_U = \sum_{\lambda \in \chi^{-1}(B) \cap U} i_\lambda(\chi)$. Hence we just need to prove that the map $\chi|_U$ is proper.

Since \mathbb{C} is a metric space, for all $\lambda \in \Lambda$ there exists a compact neighborhood C of λ such that $C \subset \Lambda$. Let C be a compact neighborhood of λ such that $\{\lambda\} = C \cap \chi^{-1}(B)$. Since C is compact, $\chi : C \rightarrow K = \chi(C)$ is proper, and since χ is continuous, K is compact. The set C is a neighborhood of λ , then $\lambda \notin \partial C$, and since $\{\lambda\} = C \cap \chi^{-1}(B)$, $B \cap \chi(\partial C) = \emptyset$. Since $i_\lambda(\chi) > 0$ we can assume, taking C small if necessary, that $\text{Ind}_{\chi(\partial C)}(B) = i_\lambda(\chi) > 0$, so that $\mathbb{C} \setminus \chi(\partial C)$ has a bounded connected component homeomorphic to a disc containing B . Therefore, there exists V open neighborhood of B homeomorphic to a disc such that $V \cap \chi(\partial C) = \emptyset$. Let U be the connected component of $\chi^{-1}(V)$ containing λ . Then $\chi^{-1}(V) \cap \partial C = \emptyset$, hence $U \cap \partial C = \emptyset$, and since $\lambda \in U$, $U \subset C$. Therefore the map $\chi|_U : U \rightarrow V$ is open and proper of degree $d = i_\lambda(\chi)$.

2. By the lifting criterion, (cfr. [H] prop 1.33 pag. 60), if $p : (X, x_0) \rightarrow (Y, y_0)$ is a map with X path connected and locally path connected, and $\pi : (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$ is a covering space, then a lift $\tilde{p} : (X, x_0) \rightarrow (\tilde{Y}, \tilde{y}_0)$ exists if and only if $p_*(\pi_1(X, x_0)) \subseteq \pi_*(\pi_1(\tilde{Y}, \tilde{y}_0))$. Then we need $\chi_*(\pi_1(U \setminus \{\lambda\})) \subseteq \pi_*(\pi_1(\tilde{V} \setminus \{\tilde{B}\}))$. Note that, by (1), χ induces a proper surjective map between U and V of degree $d = i_\lambda(\chi)$. Hence, since $\pi_1(U \setminus \{\lambda\}) = \mathbb{Z} = \pi_1(V \setminus \{B\})$, the mapping $\chi_* : \pi_1(U \setminus \{\lambda\}) \rightarrow \pi_1(V \setminus \{B\})$ is multiplication by the integer $d = i_\lambda(\chi)$. Similarly, $\pi_1(\tilde{V} \setminus \{\tilde{B}\}) = \mathbb{Z}$ and the map $\pi_* : \pi_1(\tilde{V} \setminus \{\tilde{B}\}) \rightarrow \pi_1(V \setminus \{B\})$ is multiplication by the integer d , since $\pi : \tilde{V} \rightarrow V$ is the projection of the d -folder cover of V . Therefore $\chi_*(\pi_1(U \setminus \{\lambda\})) = d\mathbb{Z} = \pi_*(\pi_1(\tilde{V} \setminus \{\tilde{B}\}))$, and finally there exists a lift of χ to π . By openness \tilde{f} is a homeomorphism, and then if $d = 1$ the map χ restricts to a homeomorphism $\chi : U \rightarrow V$.

□

Corollary 4.1. *In the notation of the above Proposition, the critical points of χ , i.e. the points of Λ where $i_\lambda(\chi) > 1$, form a closed discrete subset of Λ .*

Proof. Suppose $\lambda \in U \cap \Lambda$ is a critical point. If $q \in U \cap \Lambda$ and $q \neq \lambda$, then $i_q(\chi) = i_q(\tilde{f}) = 1$ (since \tilde{f} is a homeomorphism and π is ramified only above B). Indeed by prop. 4.5, $\chi(q) = \pi \circ \tilde{f} = n \neq B$, and since π is a covering branched at B , there exists a neighborhood $U(n)$ in V such that $\pi^{-1}(U(n)) = \bigcup_{j \in J} U_j$, and all mappings $\pi|_{U_j} : U_j \rightarrow U(n)$ are homeomorphisms. In particular $i_{\tilde{f}(q)}(\pi) = 1$ and thus $i_q(\chi) = i_q(\tilde{f})i_{\tilde{f}(q)}(\pi) = i_q(\tilde{f})$. Trivially, this set is closed since its complement (the set of points of Λ where $i_\lambda(\chi) = 1$) is an open set (indeed if $\lambda' \in \Lambda$ has $i_{\lambda'}(\chi) = 1$, then there exists a neighborhood $U(\lambda')$ of λ' such that $\forall z \in U(\lambda')$, $i_z(\chi) = 1$). Hence λ is the only critical point in $U \cap \Lambda$. \square

Proposition 4.6. *Let M be a closed and connected subset of \mathcal{B} , and $P = \chi^{-1}(M)$. If P is compact, then χ restricts to a proper map $\chi : P \rightarrow M$, and:*

1. *For any $m \in M$, set $d_m = \sum_{p \in \chi^{-1}(m)} i_p(\chi)$. The number $d = d_m$ is independent of the choice of $m \in M$.*
2. *There exist neighborhoods \hat{V} of M in Y and \hat{U} of P in X such that χ restricts to a proper mapping $\chi : \hat{U} \rightarrow \hat{V}$ of degree d .*

Proof. 2. The proof follows the one in the context of topological holomorphy of [DH].

Since P is compact, $P \subset \Lambda \subseteq \mathbb{C}$, and $P \cap \partial\Lambda = \emptyset$, the distance $r = \text{dist}(P, \partial\Lambda)$ is positive. Let N be a closed neighborhood of P in Λ with $\text{dist}(P, \partial N) = r/2 = \text{dist}(N, \partial\Lambda)$. Hence $P \subset N \subset \Lambda$, and $\chi : N \rightarrow \chi(N)$ is proper. Since $P = \chi^{-1}(M)$, and $\partial P \cap \partial N = \emptyset$, $\partial M \cap \chi(\partial N) = \emptyset$. Call \hat{V} the connected component of $\mathcal{B} \setminus \chi(\partial N)$ which contains M , and set $\hat{U} = \chi^{-1}(\hat{V}) \cap N = \chi^{-1}(\hat{V}) \cap \overset{\circ}{N}$. Then $\chi^{-1}(\hat{V}) \cap \partial N = \emptyset$, hence $\hat{U} \subset N$. Therefore the map $\chi|_{\hat{U}} : \hat{U} \rightarrow \hat{V}$ is proper. The map χ is continuous, hence, since \hat{V} is connected, \hat{U} is the union of connected components. Let us set $\hat{U} = \bigcup_j \hat{U}_j$. The restriction $\chi : \hat{U}_j \rightarrow \hat{V}$ is then a proper map between connected sets, thus it has a degree, which we call d_j . Note that, for all j , $d_j > 0$. Therefore $\chi : \hat{U} \rightarrow \hat{V}$ has a degree:

$$d = \deg \chi|_{\hat{U}} = \sum_j d_j$$

Moreover, since \hat{U} , \hat{V} open, $\chi : \hat{U} \rightarrow \hat{V}$ proper and for every $v \in \hat{V}$, $\chi^{-1}(v)$ is discrete and finite,

$$d = \deg \chi|_{\hat{U}} = \sum_{u \in \chi^{-1}(v) \cap \hat{U}} i_u(\chi).$$

1. The map $\chi : \hat{U} \rightarrow \hat{V}$ is a proper map of degree d , $M \subset \hat{V}$, $P = \chi^{-1}(M)$, and $P \subset \hat{U}$. Hence for all $m \in M$, $d = \deg \chi|_{\hat{U}} = \deg \chi|_P = \sum_{p \in \chi^{-1}(m) \cap P} i_p(\chi)$. \square

4.2 The map χ is a branched covering from a neighborhood of M_f to a neighborhood of $M_1 \setminus U(1)$.

As we saw in 1.1 and in 3.2, the range \mathcal{B} of the map χ is not the whole of \mathbb{C} , but a proper subset of \mathbb{C} , because there is no $\lambda \in \Lambda$ such that f_λ is hybrid equivalent to $P_0 = z + 1/z$. Hence $M_1 \not\subseteq \mathcal{B}$, since the root $B = 1$ does not belong to \mathcal{B} . However, we could hope that, for all $B \in \mathcal{B}$, either

1. $B \notin M_1$ as $B \rightarrow \partial\mathcal{B}$, or
2. $B \rightarrow 1$ as $B \rightarrow \partial\mathcal{B}$.

Indeed this is the case under appropriate conditions (e.g. the following one).

Definition 4.7. Let $\mathbf{f} = (f_\lambda : U'_\lambda \rightarrow U_\lambda)_{\lambda \in \Lambda}$ be an analytic family of parabolic-like maps of degree 2, such that, for $\lambda \rightarrow \partial\Lambda$:

1. $\lambda \notin M_f$ or
2. $\chi(\lambda) \rightarrow 1$.

Then we call \mathbf{f} a *nice* family of parabolic-like mappings.

If \mathbf{f} is a nice family of parabolic-like mappings, for every $U(1)$ neighborhood of the root of M_1 , setting $K = M_1 \setminus U(1)$, $C = \chi^{-1}(K)$ is compact in Λ . Indeed, if C is not compact in Λ , then there exists a sequence $(\lambda_n) \in C$ such that $\lambda_n \rightarrow \partial\Lambda$ as $n \rightarrow \infty$. On the other hand, for all n , $\chi(\lambda_n) \in K$. Let $\chi(\lambda_{n_k})$ be a subsequence converging to some parameter B . Since K is compact, the limit point B belongs to $K \subset M_1 \setminus \{1\}$. This is a contradiction, because \mathbf{f} is a nice family of parabolic-like mappings. Therefore C is compact in Λ . By prop 4.6(1) the map χ restricts to a proper map $\chi : C \rightarrow K$ of degree $\mathcal{D} > 0$, in fact by prop 4.6(2) there exist neighborhoods \hat{U} of C in Λ and \hat{V} of K in \mathcal{B} such that the restriction $\chi : \hat{U} \rightarrow \hat{V}$ is a proper map of degree \mathcal{D} . Moreover, the following theorem shows that the restriction $\chi : \hat{U} \rightarrow \hat{V}$ is a degree \mathcal{D} branched covering.

Theorem 4.8. *Given a nice family of parabolic-like maps $f_{\lambda, \lambda \in \Lambda \approx \mathbb{D}}$, the map $\chi : M_f \rightarrow M_1 \setminus \{\text{root}\}$ is a degree $\mathcal{D} > 0$ branched covering. More precisely, given $M_1 \setminus U(1) \subset K \subset \mathcal{B}$ compact and connected with $0 \in K$, there exists a \hat{V} neighborhood of K in \mathcal{B} such that the map $\chi : \hat{U} = \chi^{-1}(\hat{V}) \rightarrow \hat{V}$ is a degree $\mathcal{D} > 0$ branched covering.*

Proof. We want to prove that, for all $y \in \hat{V}$, there exists a punctured neighborhood $V^*(y)$ of y in \hat{V} such that $\chi : \chi^{-1}(V^*(y)) \rightarrow V^*(y)$ is a covering map, i.e. for all $z \in (V^*(y))$ there exists a neighborhood $V(z)$ of z in \hat{V} such that $\chi^{-1}(V(z)) = \bigcup_{j \in J} U_j$, where U_j , $j \in J$ are disjoint subsets of \hat{U} , and all mappings $\chi|_{U_j} : U_j \rightarrow V(z)$ are homeomorphisms.

By Proposition 4.6 the map $\chi : \hat{U} \rightarrow \hat{V}$ is a proper map of degree \mathcal{D} . Let $y \in \hat{V}$. By the corollary of Prop. 4.5, the set of $x \in \Lambda$ with $i_x(\chi) > 1$ is a closed discrete set, hence there exists a punctured neighborhood of $V^*(y)$ of y in \hat{V} such that, for all $x \in \chi^{-1}(V^*(y))$, $i_x(\chi) = 1$. Call $U_1^*, \dots, U_{\mathcal{D}}^*$ the preimages of $V^*(y)$.

Let $z \in V^*(y)$, and let $z_1, \dots, z_{\mathcal{D}}$ be the preimages of z in $U_1^*, \dots, U_{\mathcal{D}}^*$ respectively. Hence, by Prop. 4.5(1), for all $i \leq \mathcal{D}$ there exists neighborhoods

$U(z_i) \subset \hat{U}$ and $V_i(z) \subset \hat{V}$ of z_i and z respectively such that the map χ induces a homeomorphism $\chi : U(z_i) \rightarrow V_i(z)$.

Define $V(z) = \bigcap_i V_i(z)$, then $\chi^{-1}(V(z)) = \bigcup_{0 < i \leq \mathcal{D}} U_i$, where the U_i are disjoint subsets of \hat{U} , and all mappings $\chi|_{U_i} : U_i \rightarrow V(z)$ are homeomorphisms. \square

Corollary 4.2. *Given $B \in M_1 \setminus \{1\}$, there exists a neighborhood $V(B)$ such that $\chi^{-1}(V(B)) \rightarrow V(B)$ is a degree \mathcal{D} branched covering map.*

Proof. Let $B \in M_1 \setminus \{1\}$, and assume first that B is not a critical value for the map χ . Let $z_1, \dots, z_{\mathcal{D}}$ be the preimages of B under the map χ , and let $U(z_1), \dots, U(z_{\mathcal{D}})$ and $V_i(B)$ be neighborhoods of $z_1, \dots, z_{\mathcal{D}}$ in Λ and of $B \in M_1 \setminus \{1\}$ respectively given by Prop. 4.5(1). Set $V(B) = \bigcap_i V_i(B)$. Then $\chi^{-1}(V(B)) = \bigcup_{0 < i \leq \mathcal{D}} U_i$, where, for all $0 < i \leq \mathcal{D}$, the U_i are disjoint subsets of Λ , and the restrictions $\chi|_{U_i} : U_i \rightarrow V(B)$ are homeomorphisms.

If B is a critical value, let $V^*(B)$ be a punctured neighborhood of B in $B \in M_1 \setminus \{1\}$ such that for all $z \in V^*(B)$, $i_z(\chi) = 1$. Then, for all $z \in V^*(B)$, z is not a critical point of χ , and the result follows from the first part. \square

We call \mathcal{D} the *parametric degree* of the family \mathbf{f} , as opposed to the degree of the family, which in this case is 2. If $\mathcal{D} = 0$ then M_f is empty. If $\mathcal{D} = 1$ then χ restricts to a homeomorphism $M_f \rightarrow M_1 \setminus \{\text{root}\}$.

Call ω_λ the critical point of f_λ . Set $C = \chi^{-1}(K)$. Next proposition tells us how to compute the *parametric degree* \mathcal{D} of the family \mathbf{f} .

Proposition 4.9. *Given a nice family of parabolic-like maps $f_{\lambda, \lambda \in \Lambda \approx \mathbb{D}}$, and a compact set $M_1 \setminus U(1) \subset K \subset \mathcal{B}$, the degree \mathcal{D} of the branched covering $\chi : \hat{U} \rightarrow \hat{V}$ (where \hat{V} is a neighborhood of K in \mathcal{B} and $\hat{U} = \chi^{-1}(\hat{V})$) is equal to the number of times $f_\lambda(\omega_\lambda) - \omega_\lambda$ turns around 0 as λ describes ∂C .*

Proof. Let us remember that for every A the map $P_A = z + 1/z + A$ has two critical points: $z = 1$ and $z = -1$. After a change of coordinates we can assume $z = 1$ is the first critical point attracted by ∞ . Hence for all $A \in \mathbb{C}$, $z = -1$ is the critical point in the parabolic-like restriction of P_A (see the proof of ??).

The proof follows the one in [DH].

Let ω_λ be the critical point of f_λ . Choose λ_0 such that $f_{\lambda_0}(\omega_{\lambda_0}) = \omega_{\lambda_0}$. Let $[P_{A_0}]$ be the member of the family $Per_1(1)$ hybrid equivalent to f_{λ_0} . Therefore $P_{\pm A_0}(-1) = -1$. An easy computation shows that $\chi(\lambda_0) = B_0 = 0$. This means that the multiplicity of λ_0 as zero of $\lambda \rightarrow \chi(\lambda)$ is the multiplicity of λ_0 as zero of the map $\lambda \rightarrow f_\lambda(\omega_\lambda) - \omega_\lambda$. Hence $\mathcal{D} = \sum_{\lambda \in \chi^{-1}(0)} i_\lambda(\chi)$ is the number of zeroes of the map $\lambda \rightarrow f_\lambda(\omega_\lambda) - \omega_\lambda$ counted with multiplicity. \square

References

- [A] K. Astala, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Univ. Press, (2008).
- [Ah] L. Ahlfors, *Lectures on quasiconformal mappings*, Second edition. AMS University Lecture series, Vol. 38. (2006).

- [BF] B. Branner & N. Fagella, *Quasiconformal Surgery in Holomorphic dynamics*, preprint.
- [BH] B. Branner & J. H. Hubbard, The iteration of cubic polynomials II: Patterns and parapatterns, *Acta Math.*, 157 (1986), no. 1-2, 23–48.
- [DE] A. Douady & C. J. Earle, Conformally natural extension of homeomorphisms of the circle, *Acta Math.*, 169 (1992), no. 3-4, 229-325.
- [DH] A. Douady & J. H. Hubbard, On the Dynamics of Polynomial-like Mappings, *Ann. Sci. École Norm. Sup.*, (4), Vol.18 (1985), 287-343.
- [F] O. Forster, *Lectures on Riemann Surfaces*, Springer, (1981).
- [H] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, (2002).
- [Hö] L. Hörmander, *The analysis of Linear Partial Differential Operators I*, Springer-Verlag, (1983).
- [Hu] J. Hubbard, *Teichmüller Theory, Volume 1: Teichmüller Theory*, Matrix Editions, (2006).
- [LL] L. Lomonaco, Parabolic-like maps, arXiv:1111.7150.
- [M] J. Milnor, *Dynamics in One Complex Variable*, Annals of Mathematics Studies, (2006).
- [MSS] R. Mañé, P. Sad & D. Sullivan, On the Dynamics of Rational maps, *Ann. Sci. École Norm. Sup.*, (4), Vol.16 (1983), 193-217.
- [S] D. Sullivan, Quasiconformal Homeomorphisms and Dynamics III, *Ann. Sci. École Norm. Sup.*, (4), Vol.16 (1983), 193-217.
- [Sh] M. Shishikura, Bifurcation of parabolic fixed points, *The Mandelbrot set, Theme and Variations*, (325-363), *London Math. Soc. Lecture Note Ser.*, 274 Cambridge Univ. Press, (2000).